Fall 2025 **Math 53, Sections 104/106**

Discussion #14/15

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1. Answer the following true-or-false questions.

   (a) Any continuous function on the domain $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ will attain a maximum.
   False: $f(x, y) = x$ is a counterexample.

   (b) If $xye^x = \lambda y$ and $xye^x = \lambda x$, then we can conclude that $x = y$.
   False: It is true that $\lambda x = \lambda y$, but the case $\lambda = 0$ poses a problem. For example, if $x = 0, y = 1, \lambda = 0$, then both equations are satisfied.

   (c) If $f(x, y)$ is differentiable and attains a maximum at (a, b) in the region $\{(x, y) \in \mathbb{R}^2 : |x| + |y| \le 1\}$, then $f_x(a, b) = f_y(a, b) = 0$.
   False: This is true if (a, b) is in the interior of the region, but not necessarily if $|a| + |b| = 1$.

   (d) It is possible that a function $f(x, y)$ can have no extrema along a level curve $g(x, y) = 0$.
   True: for example $f(x, y) = x$ and $g(x, y) = y = 0$.

2. Use Lagrange multipliers to solve the following problems.

   (a) Find the extreme values of the function $f(x, y) = 2x + y + 2z$ subject to the constraint that $x^2 + y^2 + z^2 = 1$.
   We solve the Lagrange multiplier equation: $\langle 2, 1, 2\rangle = \lambda\langle 2x, 2y, 2z\rangle$. Note that λ cannot be zero in this equation, so the equalities $2 = 2\lambda x, 1 = 2\lambda y, 2 = 2\lambda z$ are equivalent to $x = z = 2y$. Substituting this into the constraint yields $4y^2 + y^2 + 4y^2 = 1$, so $y = \pm 1/3$. The max and min values occur at $(2/3, 1/3, 2/3)$ and $(-2/3, -1/3, -2/3)$, respectively, with function values ± 3.

   (b) Find the extreme values of the function $f(x, y) = y^2 e^x$ on the domain $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$.
   The gradient of this function is $(y^2 e^x, 2ye^x)$, which is zero along the x-axis $y = 0$. Here the function value of 0 is a minimum, since $f(x, y) \ge 0$ everywhere. On the boundary we have the Lagrange multiplier equation: $y^2 e^x = 2\lambda x$ and $2ye^x = 2\lambda y$. We may assume $y \ne 0$ as we have already considered this case, and then we get $2y = x/y$, so $y^2 = 2x$. Together with the equation $x^2 + y^2 = 1$, we obtain $2 - x^2 = 2x$, so $x = \pm\sqrt{2} - 1$. We only need the "+" solution because the "-" one lies outside of the unit disk. We know $y^2 = 2x = 2(\sqrt{2} - 1)$ and therefore the maximum value of f on the unit disk is

$$f\left(\sqrt{2} - 1, \pm\sqrt{2(\sqrt{2} - 1)}\right) = 2(\sqrt{2} - 1)e^{\sqrt{2} - 1}$$

(c) Use Lagrange multipliers to find the closest point(s) on the parabola $y = x^2$ to the point $(0,1)$. How could one solve this problem without using any multivariate calculus?

We maximize the function $f(x,y) = x^2 + (y-1)^2$ subject to the constraint $g(x,y) = y - x^2 = 0$. We obtain the system of equations

$$\begin{aligned} 2x &= -2\lambda x \\ 2(y-1) &= \lambda. \end{aligned}$$

Substituting the second equation into the first, we find $2x = -2(2(y-1))x$, so either $x = 0$ or $y = 1/2$. In the first case, the point $(0,0)$ is distance 1 from $(0,1)$. In the second case, $(\pm\frac{1}{\sqrt{2}}, 1/2)$ is distance $\sqrt{1/2 + 1/4} = \sqrt{3/4} < 1$ from the point $(0,1)$. These two points are the closest. (This problem could also be solved by minimizing the function $\sqrt{t^2 + (t^2-1)^2}$.)

(d) You have 24 square inches of cardboard and want to build a box (in the shape of a rectangular prism). Show that a 2″ × 2″ × 2″ cube encloses the largest volume.

If x, y, z are the side lengths of the solid, then we have a constraint $xy + yz + zx = 12$ and want to optimize the function $f(x,y,z) = xyz$. A maximum value must exist since the volume goes to zero if any of the side lengths do. We have $yz = \lambda(y+z)$ and $xz = \lambda(x+z)$ and $xy = \lambda(x+y)$. Multiplying the first equation by x and the second by y and equating, we get $x\lambda(y+z) = xyz = y\lambda(x+z)$. All quantities are positive, so we may simplify to get $x(y+z) = y(x+z)$, which simplifies to $x = y$. Arguing similarly with the third equation, we find that all side lengths are equal.

(e) Find the largest possible volume of a rectangular prism with edges parallel to the coordinate axes and all vertices lying on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

(where $a, b, c > 0$.)

Let x, y, and z each be half of the side length pointing along the coordinate axes. Then the volume of the prism is $f(x,y,z) = 8xyz$. We want to maximize this subject to the constraint $g = 1$, where $g(x,y,z) = x^2/a^2 + y^2/b^2 + z^2/c^2$. Our Lagrange multiplier equation $\nabla f = \lambda \nabla g$ becomes

$$8yz = \frac{2\lambda x}{a^2}, \qquad 8xz = \frac{2\lambda y}{b^2}, \qquad 8xy = \frac{2\lambda z}{c^2}$$

If $\lambda = 0$ then at least one of x, y, and z must be zero, giving a total volume of zero. As this is clearly not maximal, we can ignore this case and assume $\lambda \neq 0$. Multiplying the first equation by $x/2\lambda$ gives $x^2/a^2 = xyz/2\lambda$. Let $k = xyz/2\lambda$; then we are just saying $x^2/a^2 = k$. Similarly, we obtain $y^2/b^2 = z^2/c^2 = k$. Plugging these into the equation for the ellipse gives $3k = 1$, so $k = 1/3$. Thus $x = \pm\frac{1}{\sqrt{3}}a$, and since x is a length, we should get $x = \frac{1}{\sqrt{3}}a$. Similarly, we obtain $y = \frac{1}{\sqrt{3}}b$ and $z = \frac{1}{\sqrt{3}}c$.

(f) Use Lagrange multipliers to find the closest points to the origin on the hyperbola $xy = 1$.

We want to minimize $f(x,y) = x^2 + y^2$ subject to $g(x,y) = 1$, where $g(x,y) = xy$. Setting $\nabla f = \lambda \nabla y$, we obtain $2x = \lambda y$ and $2y = \lambda x$. If $\lambda = 0$, then $x = y = 0$, but $(0,0)$ is not a point on the hyperbola, so we can ignore this case. So $\lambda \neq 0$, and we can write $y = 2x/\lambda$. Plugging this into $xy = 1$, we get $2x^2/\lambda = 1$, or $\lambda = 2x^2$. Taking this equation and plugging it into $2y = \lambda x$, we see $2y = 2x^3$, or $y = x^3$. Then $1 = xy = x^4$, so $x = \pm 1$. For $x = 1$ we solve $xy = 1$ to get $y = 1$; likewise, for $x = -1$ we get $y = -1$. It is geometrically obvious that these correspond to minima, so the closest points to the origin on $xy = 1$ are $(1,1)$ and $(1,-1)$.

3. Here are some more Lagrange multiplier problems.

(a) Consider the functions $f(x,y,z) = x + 4y + 4z$, $g(x,y,z) = x^2 + 4y^2 + 4z^2$.

i. $g(x,y,z) = 2$ parameterizes an ellipsoid. Find the maximum and minimum of f on the ellipsoid given by $g(x,y,z) = 2$.

We use the Lagrange multiplier for this question.

$$\nabla f = \lambda \nabla g$$
$$(1,4,4) = \lambda(2x, 8y, 8z)$$

First of all, $\lambda \neq 0$. Then it is fairly obvious that to satisfy the equation, we must have $x = y = z$. So we solve for $g(x,x,x) = 9x^2 = 2$. This gives

$$x = y = z = \pm\frac{\sqrt{2}}{3}$$

We plug in the points to find that

$$f\left(\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}\right) = 3\sqrt{2}; \quad f\left(\frac{-\sqrt{2}}{3}, \frac{-\sqrt{2}}{3}, \frac{-\sqrt{2}}{3}\right) = -3\sqrt{2}$$

Of which $3\sqrt{2}$ is the maximum value of f on the ellipsoid, and $-3\sqrt{2}$ is the minimum.

ii. What is the maximum and minimum of f among the points satisfying $g(x,y,z) \leqslant 2$?

Since $\nabla f \neq 0$, there is no critical points inside the ellipsoid, and so the maximum and minimum is the same as the ones in **b.**, $\pm 3\sqrt{2}$.

(b) Consider the function $f(x,y,z) = xy + xz + yz$.

i. What is the maximum and minimum of f on the sphere $g(x,y,z) = \frac{1}{2}(x^2 + y^2 + z^2) = 2$.

To solve the problem, we need to use the Lagrange multiplier

$$\nabla f = \lambda \nabla g$$
$$(y+z, x+z, x+y) = \lambda(x,y,z)$$

This gives me three equations:

$$\lambda x = y + z; \qquad \lambda y = x + z; \qquad \lambda z = x + y$$

Observe that if we add all three equations,

$$\lambda(x+y+z) = 2(x+y+z) \tag{0.1}$$

Then $\lambda = 2$ or $x+y+z = 0$, which correspond to $\lambda = -1$ For $\lambda_1 = 2$, we can plug everything back in and see that $x = y = z$, then we must have $x = y = z = \pm\dfrac{2}{\sqrt{3}}$.

$$f\left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) = 4 \qquad f\left(-\frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right) = 4$$

For $\lambda_2 = -1$, we see that $x+y+z = 0$. However, fear not that you can not solve for a point. Consider

$$(x+y+z)^2 = x^2+y^2+z^2+2xy+2xz+2yz = 4+2f(x,y,z) = 0$$

Which means that $f(x,y,z) = -2$ on all points such that $x+y+z = 0$ and $x^2+y^2+z^2 = 4$. Then we see the maximum of f on the sphere is 4 while the minimum is -2.

ii. What is the maximum and minimum of f inside the solid sphere including the boundary $g(x,y,z) \leqslant 2$.
We need to find critical point of the function inside the sphere, so we set $\nabla f = (0,0,0)$, this gives me three equations:

$$y+z = 0 \qquad x+z = 0 \qquad x+y = 0$$

We observe that

$$(y+z)-(x+z)-(x+y) = -2x = 0$$
$$-(y+z)+(x+z)-(x+y) = -2y = 0$$
$$+(y+z)-(x+z)+(x+y) = -2z = 0$$

So f has only one critical point at $(0,0,0)$, and $f(0,0,0) = 0$. But $-2 < 0 < 4$, so the maximum and minimum is still the ones we found in **a.**, 4 and -2.

4. Use Lagrange multipliers to solve the following problems.

(a) Maximize and minimize $3x-y-3z$ subject to $x+y-z = 1$ and $x^2+2z^2 = 1$. Let $f = 3x-y-3z$, $g = x+y-z$, $h = x^2+2z^2$. Then $\nabla f = (3,-1,-3)$, $\nabla g = (1,1,-1)$, and $\nabla h = (2x,0,4z)$. Our Lagrange multiplier equation $\nabla f = \lambda\nabla g + \mu\nabla h$ splits into

$$3 = \lambda + 2\mu x, \qquad -1 = \lambda + 0, \qquad -3 = -\lambda + 4\mu z.$$

Hence $\lambda = -1$, and we can plug this in to the other equations to see $\mu = 2/x = -1/z$, so $x = -2z$. Plugging this into $x^2+2z^2 = 1$ gives $6z^2 = 1$ so $z = \pm 1/\sqrt{6}$, $x = \mp 2/\sqrt{6}$ (so x has the opposite sign of z). Plugging this into $x+y-z = 1$ shows $y = 1+3z$ and so (x,y,z) is either

$$(-2/\sqrt{6}, 1+3/\sqrt{6}, 1/\sqrt{6})$$

or

$$(2/\sqrt{6}, 1-3/\sqrt{6}, -1/\sqrt{6}).$$

Computing $3x - y - 3z$ for each shows that the former gives a minimum $(-1 - 2\sqrt{6})$ and the latter gives a minimum $(1 + 2\sqrt{6})$.

(b) Maximize and minimize z subject to $x^2 + y^2 = z^2$ and $x + y + z = 24$. This has no maximum or minimum. How do we see this? We show that when z is large (how large exactly we're about to see) then the system

$$x^2 + y^2 = z^2$$
$$x + y + z = 24$$

has a solution (x, y). To check this we solve for y in the second equation and plug back into the first, obtaining

$$x^2 + (24 - x - z)^2 = z^2$$

which simplifies to

$$x^2 + (z - 24)x + (288 - 24z) = 0$$

This is a quadratic equation and we know that they have solutions when the discriminant is greater or equal to zero. Here the discriminant is

$$(z - 24)^2 + 4(288 - 24z) = z^2 + 48z - 576$$

This describes a parabola that's "open from above" so when z is very large or very negative the discriminant will be positive, meaning that there are x, y such that (x, y, z) satisfies our constraints. So $f(x, y, z)$ can be arbitrarily large and arbitrarily small given our constraints.

5. Here are some challenge problems.

(a) Using the method of Lagrange multipliers, prove the following inequality: if $x_1, \ldots, x_n$ are positive real numbers, then

$$\frac{n}{1/x_1 + \cdots + 1/x_n} \leq \sqrt[n]{x_1 \ldots x_n}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$. The left-hand side is called the *harmonic mean* of the numbers $x_1, \ldots, x_n$ and the right-hand side is called their *geometric mean*, and this result is known as the *GM–HM inequality*.

We maximize the function $f(x_1, \ldots, x_n) = \frac{n}{1/x_1 + \cdots + 1/x_n} \leq \sqrt[n]{x_1 \ldots x_n}$ subject to the constraint that $g(x_1, \ldots, x_n) := x_1 \ldots x_n = C$ for a constant C. Note that maximizing f is equivalent to minimizing the function $F(x_1, \ldots, x_n) = \frac{1}{x_1} + \cdots + \frac{1}{x_n}$. This function must obtain a minimum on the hypersurface $x_1 \ldots x_n = C > 0$ because this quantity tends to infinity as $\min(x_1, \ldots, x_n) \to 0$, so the minimum must occur at a point found by Lagrange multipliers (since the gradient of the constraint function is nonzero on its level set.) For each k, we have

$$\frac{-1}{x_k^2} = \lambda x_1 \ldots \hat{x}_k \ldots x_n.$$

Where the hat over x_k indicates that it is omitted from the product. Rearranging,

$$-1 = \lambda x_1 \ldots x_k^2 \ldots x_n = C\lambda x_k.$$

Now, λ must be nonzero for this to hold, in which case we find that $x_1 = \cdots = x_n$ $(= \sqrt[n]{C})$, which we may check gives equality for the claimed inequality. By the previous reasoning, this must correspond to a minimum for F, or a maximum for f, so at any other point, the LHS is strictly smaller than the RHS.

(b) If $x_1, \ldots, x_n$ are real numbers, prove that

$$\frac{1}{n}\sum_{i=1}^{n} x_i \leq \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2}.$$

The left-hand side is called the *arithmetic mean* of the numbers $x_1, \ldots, x_n$ and the right-hand side is called their *quadratic mean*, and this result is known as the *QM–AM inequality*.

Let $r = \sqrt{\sum_i x_i^2}$. Define functions $f(y_1, \ldots, y_n) = \sum_i y_i$ and $g(y_1, \ldots, y_n) = \sum_{i=1}^n y_i^2$. To show our desired inequality, it suffices to show that the maximum value of f on the sphere $g(y_1, \ldots, y_n) = r$ is at most $\sqrt{nr}$ (because then $f(x_1, \ldots, x_n) \leq \sqrt{nr}$, so $f(x_1, \ldots, x_n)^2 \leq nr$, which is exactly the inequality we are trying to show). So we optimize f subject to the constraint $g = r$. To do this, we use Lagrange multipliers, and so we set $\nabla f(y_1, \ldots, y_n) = \lambda \nabla g(y_1, \ldots, y_n)$ for some scalar λ. Computing our gradients and plugging them in, we get $1 = 2\lambda y_i$ for each i. Thus we must have $y_i = 1/(2\lambda)$ for all i (since $\lambda = 0$ would lead to the equation $1 = 0$, which can't hold). Plugging these into the equation $g(y_1, \ldots, y_n) = r$, we obtain

$$r = \sum_{i=1}^{n} \frac{1}{4\lambda^2} = \frac{n}{4\lambda^2},$$

so $\lambda = \pm\frac{1}{2}\sqrt{n/r}$. It follows that $y_i = \pm\sqrt{r/n}$ for all i, so

$$\sum_i y_i = n \cdot \pm\sqrt{\frac{r}{n}} = \pm\sqrt{nr}.$$

The (global) maximum is clearly obtained when the sign here is $+$, so we see that the maximum value of f on the sphere $g = r$ is $\sqrt{nr}$, as needed.

Problem 3 courtesy of Galen Liang. All other problems courtesy of Carlos Esparza.