**Fall 2025** **Math 53, Sections 104/106**

# Discussion #14/15

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1. Answer the following true-or-false questions.
   (a) Any continuous function on the domain $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ will attain a maximum.
   (b) If $xye^x = \lambda y$ and $xye^x = \lambda x$, then we can conclude that $x = y$.
   (c) If $f(x, y)$ is differentiable and attains a maximum at $(a, b)$ in the region $\{(x, y) \in \mathbb{R}^2 : |x| + |y| \leq 1\}$, then $f_x(a, b) = f_y(a, b) = 0$.
   (d) It is possible that a function $f(x, y)$ can have no extrema along a level curve $g(x, y) = 0$.

2. Use Lagrange multipliers to solve the following problems.
   (a) Find the extreme values of the function $f(x, y) = 2x + y + 2z$ subject to the constraint that $x^2 + y^2 + z^2 = 1$.
   (b) Find the extreme values of the function $f(x, y) = y^2e^x$ on the domain $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.
   (c) Use Lagrange multipliers to find the closest point(s) on the parabola $y = x^2$ to the point $(0, 1)$. How could one solve this problem without using any multivariate calculus?
   (d) You have 24 square inches of cardboard and want to build a box (in the shape of a rectangular prism). Show that a $2'' \times 2'' \times 2''$ cube encloses the largest volume.
   (e) Find the largest possible volume of a rectangular prism with edges parallel to the coordinate axes and all vertices lying on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

   (where $a, b, c > 0$.)
   (f) Use Lagrange multipliers to find the closest points to the origin on the hyperbola $xy = 1$.

3. Here are some more Lagrange multiplier problems.

   (a) Consider the functions $f(x,y,z) = x + 4y + 4z$, $g(x,y,z) = x^2 + 4y^2 + 4z^2$.

      i. $g(x,y,z) = 2$ parameterizes an ellipsoid. Find the maximum and minimum of $f$ on the ellipsoid given by $g(x,y,z) = 2$.

      ii. What is the maximum and minimum of $f$ among the points satisfying $g(x,y,z) \leqslant 2$?

   (b) Consider the function $f(x,y,z) = xy + xz + yz$.

      i. What is the maximum and minimum of $f$ on the sphere $g(x,y,z) = \frac{1}{2}(x^2 + y^2 + z^2) = 2$.

      ii. What is the maximum and minimum of $f$ inside the solid sphere including the boundary $g(x,y,z) \leqslant 2$.

4. Use Lagrange multipliers to solve the following problems.

   (a) Maximize and minimize $3x - y - 3z$ subject to $x + y - z = 1$ and $x^2 + 2z^2 = 1$.

   (b) Maximize and minimize $z$ subject to $x^2 + y^2 = z^2$ and $x + y + z = 24$.

5. Here are some challenge problems.

   (a) Using the method of Lagrange multipliers, prove the following inequality: if $x_1, \ldots, x_n$ are positive real numbers, then

   $$\frac{n}{1/x_1 + \cdots + 1/x_n} \leq \sqrt[n]{x_1 \ldots x_n}$$

   with equality if and only if $x_1 = x_2 = \cdots = x_n$. The left-hand side is called the *harmonic mean* of the numbers $x_1, \ldots, x_n$ and the right-hand side is called their *geometric mean*, and this result is known as the *GM–HM inequality*.

   (b) If $x_1, \ldots, x_n$ are real numbers, prove that

   $$\frac{1}{n}\sum_{i=1}^{n} x_i \leq \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2}.$$

   The left-hand side is called the *arithmetic mean* of the numbers $x_1, \ldots, x_n$ and the right-hand side is called their *quadratic mean*, and this result is known as the *QM–AM inequality*.