Fall 2025 **Math 53, Sections 104/106**

Discussion #11

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Date: September 26

1. What are the extreme values of $f(x,y) = x^2 - 10xy + y^2$ on all of $\mathbb{R}^2$? Try to find its extrema using the gradient vector and the discriminant D. Think about the answer in relation to the values of f_{xx} and f_{yy}.

   Firstly, the function grows unboundedly to $\pm\infty$. If $x = -y$, then $f(x, -x) = 8x^2 \longrightarrow \infty$; if $x = y$ then $f(x,x) = -8x^2 \longrightarrow -\infty$. However, if we try to solve this with partials, then computing $\nabla f(x,y) = \langle 2x - 10y, -10x + 2y\rangle$ and setting to $\langle 0, 0\rangle$ gives that the only potential extremum is $(0,0)$. Here, the Hessian is $\begin{pmatrix} 2 & -10 \\ -10 & 2 \end{pmatrix}$, so $D = -96$. This tells us that the origin is a saddle point. In particular, the origin is not a minimizer, even though $f_{xx} = f_{yy} = 2 > 0$. If we were to change coordinates to $u = \frac{1}{\sqrt{2}}(x+y)$ and $v = \frac{1}{\sqrt{2}}(x-y)$, and define $g(u,v) = f(x,y)$, then we would find in fact that $g_{uu} < 0$ and $g_{vv} > 0$.

2. Find the local maximum and minimum values and saddle point(s) of the following functions.

   (a) $f(x,y) = x^2 + y^4 + 2xy$
   (b) $f(x,y) = xy + e^{-xy}$
   (c) $f(x,y) = y\sin(\pi x)$

   (a) We have $f_x = 2x + 2y, f_y = 4y^3 + 2x, f_{xx} = f_{xy} = 2, f_{yy} = 12y^2$. Then $f_x = 0$ implies $y = -x$ and substituting into f_y yields $4y^3 - 2y = 0$. Either $y = 0$ or $y = \pm 1/\sqrt{2}$ so the critical points are $(0,0), (1/\sqrt{2}, -1/\sqrt{2}), (-1/\sqrt{2}, 1/\sqrt{2})$. Now $D(x,y) = 2(12y^2) - 2^2 = 24y^2 - 4$. $D(0,0) = -4 < 0$ so $(0,0)$ is a saddle point. $D(1/\sqrt{2}, -1/\sqrt{2}) = D(-1/\sqrt{2}, 1/\sqrt{2}) = 12 - 4 = 8 > 0$ and $f_{xx} = 2 > 0$ so both points correspond to a local minima.

   (b) We have $f_x = y - ye^{-xy}, f_y = x - xe^{-xy}, f_{xx} = y^2e^{-xy}, f_{xy} = 1 + (xy-1)e^{-xy}, f_{yy} = x^2e^{-xy}$. Then $f_x = 0$ implies $y(1 - e^{-xy}) = 0$ so either $y = 0$ or $x = 0$. If $x = 0$, then $f_y = 0$ for all y so all points of the form $(0, y_0)$ are critical points. If $y = 0$, $f_y = 0$ for all x values so any point of the form $(x_0, 0)$ is a critical point. We have $D(x_0, 0) = 0 = D(0, y_0)$ so the Second Derivative Test gives us no information. If we let $t = xy$ then $f(x,y) = g(t) = t + e^{-t}$. Then $g'(t) = 1 - e^{-t}$. Then $g'(t) = 0$ only for $t = 0$ and $g''(0) = 1 > 0$ so $g(0) = 1$ is a local minimum. It is an absolute minimum because $g'(t) < 0$ for $t < 0$ and $g'(t) > 0$ for $t > 0$. Thus, $f(x,y) = xy + e^{-xy} \geq 1$ for all (x,y) with equality iff $x = 0$ or $y = 0$. Hence, all the critical points we found correspond to local (and absolute) minima.

(c) $f_x = \pi y \cos(\pi x)$ and $f_y = \sin(\pi x)$, so to have $\nabla f = \langle 0, 0 \rangle$ we need $x \in \mathbb{Z}$ and there $f_x = \pi y$, so $y = 0$; i.e., the critical points are of the form $(n, 0)$. We now compute the Hessian: $f_{xx} = -\pi^2 y \sin(\pi x)$, $f_{xy} = \pi \cos(\pi x)$, and $f_{yy} = 0$. Its determinant is $-\pi^2 \cos^2(\pi x) < 0$, so all of the critical points are saddle points.

3. Find the global maxima and minima of the following functions on their indicated domains.

   (a) The function $f(x, y) = x^2 - y$ on the domain $D = [0, 2] \times [0, 2]$.

   (b) The function $f(x, y) = x - y$ on the domain $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.

   (c) The function $f(x, y) = x^2 - xy + y^2 - 3y$ on the region bounded by the x and y axes and the line $x + y = 4$.

   (d) The function $f(x, y) = \sin x \cdot \sin y$ on the domain $-1 < x < \pi$ and $-\pi < y < \pi$.

(a) On the domain, x^2 is maximized at $x = 2$ and y is minimized at $y = 0$, so f is maximized at the point $(2, 0)$ (with value 4) and similarly f is minimized at $(0, 2)$, with value -2.

(b) The gradient vector $\langle 1, -1 \rangle$ is never zero, so all extrema lie on the boundary. We parameterize the boundary $x = \cos(t), y = \sin(t)$, so that $f(\cos(t), \sin(t)) = \cos(t) - \sin(t)$. The derivative of this function is $-\sin(t) - \cos(t)$, which vanishes at $t = 3\pi/4$ and $t = 7\pi/4$. These correspond to minima and maxima, respectively.

(c) The gradient vector is $\langle 2x - y, -x + 2y + 1$, which vanishes only at the point $(1, 2)$, which is in the interior of the domain. We see $f(1, 2) = -3$. As this is the only critical point on all of $\mathbb{R}^2$ and it is a minimum (for example by the second derivative test), it is a global minimum. We examine the three boundary lines to look for possible maxima. On the x-axis with $0 \leq x \leq 4$, we have $f(x, 0) = x^2$, which is maximized at $f(4, 0) = 16$. On the y-axis, with $0 \leq y \leq 4$, we have $f(0, y) = y^2 - 3y$, which is maximized at $f(0, 4) = 4$. Finally, on the line $x = 4 - y$ with $0 \leq y \leq 4$, we have $f(4 - y, y) = 3y^2 - 15y + 16$, which is maximized at $f(4, 0) = 16$. So, altogether, there is a minimum at $f(1, 2) = -3$ and a maximum at $f(4, 0) = 16$.

(d) $\nabla f = \langle \cos x \cdot \sin y, \sin x \cdot \cos y \rangle$. If we want to make $f_x = 0$ by making $\cos x = 0$, then we need $x = \frac{\pi}{2}$ (note! *not* $-\frac{\pi}{2}$, which means to make $f_y = 0$ we need $\cos y = 0$, which means $y = \pm\frac{\pi}{2}$. Otherwise, we have to make $f_x = 0$ by making $\sin y = 0$, so $y = 0$, and then $f_y = 0$ by $\sin x = 0$, so $x = 0$. Thus we have 3 critical points to worry about. The Hessian is $\begin{pmatrix} -\sin x \cdot \sin y & \cos x \cdot \cos y \\ \cos x \cdot \cos y & -\sin x \cdot \sin y \end{pmatrix}$ which has determinant $\sin^2 x \cdot \sin^2 y - \cos^2 x \cdot \cos^2 y$. The best thing to do now is just plug in our values: $(\frac{\pi}{2}, -\frac{\pi}{2})$ has $D = 1$ by $f_{xx} = f_{yy} > 0$, so it's a local minimum. $(\frac{\pi}{2}, \frac{\pi}{2})$ has $D = 1$ by $f_{xx} = f_{yy} < 0$, so it's a local maximum. $(0, 0)$ has $D = -1$ but $f_{xx} = f_{yy} = 0$, so it's a saddle point.

Problems 2(a–b) and 3(a–c) courtesy of Carlos Esparza. Problems 2(c) and 3(d) courtesy of Galen Liang.