Fall 2025 **Math 53, Sections 104/106**

Discussion #9

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1. Do directional derivatives commute? i.e., for unit vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, and twice-partially-differentiable f (with any number of inputs; you can assume 2), is it the case that $D_{\boldsymbol{v}} D_{\boldsymbol{u}} f = D_{\boldsymbol{u}} D_{\boldsymbol{v}} f$? Either prove it or provide a counterexample.

   We can just directly evaluate the left-hand side. Say $\boldsymbol{u} = (u_1, u_2)$ and $\boldsymbol{v} = (v_1, v_2)$. Then $D_{\boldsymbol{u}} f = u_1 f_x + u_2 f_y$ so $D_{\boldsymbol{v}} D_{\boldsymbol{u}} f = v_1(u_1 f_{xx} + u_2 f_{xy}) + v_2(u_1 f_{xy} + u_2 f_{yy}) = u_1 v_1 f_{xx} + u_2 v_2 f_{yy} + (u_1 v_2 + u_2 v_1) f_{xy}$, so if we swap the roles of $\boldsymbol{u}$ and $\boldsymbol{v}$ we get the same outcome.

2. Suppose the following are true:

$$D_{\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle} f = e^x(\sin y + \cos y)$$
$$D_{\left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle} f = e^x(-\sin y + \cos y).$$

   Find ∇f.

   Call the directions $\boldsymbol{u}$ and $\boldsymbol{v}$. Then $\boldsymbol{u} + \boldsymbol{v} = \sqrt{2}\boldsymbol{i}$ and $\boldsymbol{u} - \boldsymbol{v} = \sqrt{2}\boldsymbol{j}$. Since we want $D_{\boldsymbol{i}} f$ and $D_{\boldsymbol{j}} f$, the answer is

$$\left\langle \sqrt{2} e^x \cos y, \sqrt{2} e^x \sin y \right\rangle.$$

3. Compute the following tangent planes:

   (a) $f(x, y, z) = d$, for $f(x, y, z) = ax + by + cz$, at any point (x_0, y_0, z_0). (Do this one with partial derivatives. Could you have done this another way?)

   (b) $xy^2z^3 = 8$ at $(2, 2, 1)$;

   (c) $x + y + z = e^{xyz}$ at $(0, 0, 1)$.

   (d) Show that the equation of the tangent plane to the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ at the point (x_0, y_0, z_0) can be written as

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = 1.$$

   (a) $\nabla f = \langle a, b, c \rangle$, so from the formula the tangent plane is $a(x - x_0) + b(y - y_0) + c(z - z_0) = d$, which is exactly the original plane. Of course, the plane is tangent to itself!

   (b) Let $F(x, y, z) = xy^2z^3$. Then $\nabla F(x, y, z) = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$ so $\nabla F(2, 2, 1) = \langle 4, 8, 24 \rangle$. Hence, the tangent plane is $4(x - 2) + 8(y - 2) + 24(z - 1) = 0$.

(c) Let $F(x,y,z) = x+y+z-e^{xyz}$. Then $\nabla F(x,y,z) = \langle 1-yze^{xyz}, 1-xze^{xyz}, 1-xye^{xyz}\rangle$ so $\nabla F(0,0,1) = \langle 1,1,1\rangle$. Hence, the tangent plane is $(x-0)+(y-0)+(z-1)=0$.

(d) $\nabla F(x_0,y_0,z_0) = \langle 2x_0/a^2, 2y_0/b^2, 2z_0/c^2\rangle$. Then the tangent plane is

$$\frac{2x_0}{a^2}(x-x_0)+\frac{2y_0}{b^2}(y-y_0)+\frac{2z_0}{c^2}(z-z_0)=0.$$

Rearranging, we obtain

$$\frac{2x_0}{a^2}x+\frac{2y_0}{b^2}y+\frac{2z_0}{c^2}z = 2\left(\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}+\frac{z_0^2}{c^2}\right)=2.$$

Dividing by 2 gives the desired result.

4. (*This one is pretty tricky*) Consider the spheres $\left(x-\frac{1}{\sqrt{2}}\right)^2+y^2+z^2=1$ and $\left(x+\frac{1}{\sqrt{2}}\right)^2+y^2+z^2=1$. For each point of their intersection, find the angle between the tangent plane to each sphere.

What we can immediately do is compute the tangent plane to each point (x_0,y_0,z_0) of intersection. It's just a matter of computing the respective tangent planes: for the first circle, $2\left(x_0-\frac{1}{\sqrt{2}}\right)(x-x_0)+2y_0(y-y_0)+2z_0(z-z_0)=a_{x_0,y_0,z_0}$ for some number a_{x_0,y_0,z_0}. Similarly, for the second circle, $2\left(x_0+\frac{1}{\sqrt{2}}\right)(x-x_0)+2y_0(y-y_0)+2z_0(z-z_0)=b_{x_0,y_0,z_0}$ for some number b_{x_0,y_0,z_0}. The angle is given by the angle between the normal vectors, which have dot product $x_0^2-\frac{1}{2}+y_0^2+z_0^2$. So we need to know the value of $x_0^2+y_0^2+z_0^2$. We can now take the sum of the two defining equations at the mutual point (x_0,y_0,z_0), to get $2(x_0^2+y_0^2+z_0^2)+1=2$, or $x_0^2+y_0^2+z_0^2=\frac{1}{2}$. Thus the dot product is always 0, i.e. the angle is always $90°$!

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