Fall 2025 **Math 53, Sections 104/106**

Discussion #7

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1. For each of the following, determine whether the limit exists.

   (a) $f(x,y) = \alpha x + \beta y + \gamma$ and $(x,y) \to (a,b)$. (Try doing this one with ε–δ.)

   (b) $f(x,y) = xy \sin \dfrac{1}{x^2+y^2}$ and $(x,y) \to (0,0)$.

   (c) $f(x,y) = \dfrac{(x-y)^2(x+y)}{(x-y)^4+(x+y)^2}$ and $(x,y) \to (0,0)$.

   (a) The answer should be $\ell = \alpha a + \beta b + \gamma$. Let's prove it. Given any $\varepsilon > 0$, we want to show there is some δ such that if (x,y) are within δ of (a,b), then $f(x,y)$ is within ε of ℓ. We compute that $|f(x,y) - \ell| = |\alpha(x-a) + \beta(y-b)|$. By the triangle inequality, this is at most $|\alpha||x-a| + |\beta||x-b|$. We've seen the trick $|w| \leqslant \sqrt{w^2+z^2}$ come up, and we use it again: $|x-a|, |y-b| \leqslant \sqrt{(x-a)^2+(y-b)^2}$, so $|f(x,y)-\ell| \leqslant (|\alpha|+|\beta|)\sqrt{(x-a)^2+(y-b)^2}$. The condition that (x,y) is within δ of (a,b) is to say that $\sqrt{(x-a)^2+(y-b)^2} < \delta$, so if $\delta = \frac{\varepsilon}{|\alpha|+|\beta|}$, then we are done. The only issue is if $\alpha = \beta = 0$, in which case $f = \gamma$ always and the limit is easy since it is a constant function.

   (b) Since $|\sin z| \leqslant 1$, we have that $|f(x,y)| \leqslant |xy|$ and we use the squeeze theorem to get that the limit is 0.

   (c) This is actually modified from §14.2, example 3. Use the substitution $u = x - y$ and $v = x + y$ (it is then still the case that $(u,v) \to (0,0)$. The point here is to give an example where the limit is in fact 0 *when only considering x and y linearly related*, but when we consider a quadratic $v = au^2$ the limit in $u \to 0$ becomes $\frac{a}{1+a^2} \neq 0$. So the limit does not exist.

2. Find the best linear approximation to each of the following functions near the corresponding input values.

   (a) $f(x,y) = y^2 - x$ near the input $(3,0)$.

   (b) $g(x,y) = e^x \cos y$ near the input $(5, \pi/2)$.

   (c) $h(x,y,z) = xyz$ near the input $(3,0,2)$.

   (d) $p(x,y,z,w) = x^2+y^2+z^2+w^2$ near the input $(0,1,0,-1)$.

(a) We have $f_x(x,y) = -1$ and $f_y(x,y) = 2y$, so $f_x(3,0) = -1$ and $f_y(3,0) = 0$. Our formula for the best linear approximation near $(3,0)$ is

$$f(x,y) \approx f(3,0) + f_x(3,0) \cdot (x-3) + f_y(3,0) \cdot (y-0),$$

so we see

$$f(x,y) \approx -3 - (x-3) = -x$$

for (x,y) near $(3,0)$.

(b) We have $g_x(x,y) = e^x \cos y$ and $g_y(x,y) = -e^x \sin y$, so $g_x(5,\pi/2) = 0$ and $g_y(5,\pi/2) = -e^5$. We also compute $g(5,\pi/2) = 0$. Plugging these into our formula for the best linear approximation gives

$$g(x,y) \approx -e^5 \left(y - \frac{\pi}{2}\right).$$

(c) We have $h_x(x,y,z) = yz$, $h_y(x,y,z) = xz$, and $h_z(x,y,z) = xy$. Thus, at the input $(3,0,2)$, we have $h = 0$, $h_x = 0$, $h_y = 6$, and $h_z = 0$. Using a formula similar to that for the 2-dimensional case, we see that the best linear approximation is

$$h(x,y,z) \approx 6(y-0) = 6y.$$

(d) Even though this is a function of four variables, our old methods still work! We just have to add a few more terms to our sums to account for the extra variables. At the input $(0,1,0,-1)$, we have $p = 2$, $p_x = 0$, $p_y = 2$, $p_z = 0$, and $p_w = -2$. Thus the best linear approximation is given by

$$p(x,y,z,w) \approx 2 + 2(y-1) - 2(w+1) = -2 + 2y - 2w.$$

3. Compute the gradients of the following functions.

   (a) $f(\theta,\phi) = \cos\theta\cos\phi$.

   (b) $f(x,y) = \arctan(y/x)$.

   (c) $f(t,x,y) = \frac{1}{\sqrt{4\pi t}} \exp(-(x-y)^2/4t))$. (Express everything as multiples of $f(t,x,y)$.)

(a)

$$\nabla f = (-\sin\theta\cos\phi, -\cos\theta\sin\phi)$$

(b)

$$\frac{\partial f}{\partial x} = \frac{-y}{x^2(1+y^2/x^2)}$$
$$\frac{\partial f}{\partial y} = \frac{1}{x(1+y^2/x^2)}$$
$$\nabla f = \frac{1}{x^2+y^2}(-y,x)$$

(c)

$$\begin{aligned}
\frac{\partial f}{\partial t} &= -\frac{1}{2\sqrt{4\pi}\,t^{3/2}} \exp\left(-\frac{(x-y)^2}{4t}\right) + \frac{1}{\sqrt{4\pi t}} \frac{(x-y)^2}{4t^2} \exp\left(-\frac{(x-y)^2}{4t}\right) \\
&= f(t,x,y)\left(\frac{(x-y)^2 - 2t}{4t^2}\right) \\
\frac{\partial f}{\partial x} &= \frac{y-x}{2t} f(t,x,y) \\
\frac{\partial f}{\partial y} &= \frac{x-y}{2t} f(t,x,y) \\
\nabla f &= \frac{f(t,x,y)}{4t^2}\left(((x-y)^2 - 2t), -2t(x-y), 2t(x-y)\right)
\end{aligned}$$

4. Consider $f(x,y) = e^{-r^4}$ where $r = \sqrt{x^2+y^2}$. Compute its directional derivative at $(0,0)$ w.r.t. the unit vectors in (polar) directions $\theta = 0, \pi/4, \pi/2$. What about any other angle?

   The function is symmetric with respect to rotations around the origin, so all the directional derivatives will be equal. Therefore it is sufficient to compute the derivative for $\theta = 0$, i.e. $\partial/\partial x$:
   $$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} e^{-(x^2+y^2)^2} = -2(x^2+y^2) \cdot 2x e^{-(x^2+y^2)^2}$$
   Plugging in $x = y = 0$ we see that this is zero.

Problems 1(b–c) courtesy of ChatGPT. Problems 2–4 courtesy of Carlos Esparza.