Fall 2025 **Math 53, Sections 104/106**

Discussion #5

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1. For each of the following functions, find their first derivatives, i.e. $f_x(x, y)$ and $f_y(x, y)$, and also compute $f_x(1, 1)$ and $f_y(1, 1)$.

   (a) $f(x, y) = x^4 + 5xy^3$

   (b) $f(x, y) = \frac{x}{y}$

   (c) $f(x, y) = x\sin(x + y) + y^2$

   (a) We can compute the derivatives as follows:

   $$f_x(x, y) = 4x^3 + 5y^3 \implies f_x(1, 1) = 9$$
   $$f_y(x, y) = 15xy^2 \implies f_y(1, 1) = 15$$

   (b) We can compute the derivatives as follows:

   $$f_x(x, y) = \frac{1}{y} \implies f_x(1, 1) = 1$$
   $$f_y(x, y) = -\frac{x}{y^2} \implies f_y(1, 1) = -1$$

   (c) We can compute the derivatives as follows:

   $$f_x(x, y) = \sin(x + y) + x\cos(x + y) \implies f_x(1, 1) = \sin(2) + \cos(2)$$
   $$f_y(x, y) = x\cos(x + y) + 2y \implies f_y(1, 1) = \cos(2) + 2$$

2. Calculate the first derivatives i.e. f_x, f_y, and f_z, for the function

   $$f(x, y, z) = \cos(x^2 + 2y) - e^{4x - z^4 y} + y^3$$

   The key thing to note here is that partial derivatives work the same no matter how many variables we take in. So we can compute the partial derivatives as follows:

   $$f_x(x, y, z) = -2x\sin(x^2 + 2y) - 4e^{4x - z^4 y}$$
   $$f_y(x, y, z) = -2\sin(x^2 + 2y) + z^4 y e^{4x - z^4 y} + 3y^2$$
   $$f_z(x, y, z) = 4z^3 y e^{4x - z^4 y}$$

3. Find the tangent planes to the graphs of each of the following functions at an arbitrary point $(x_0, y_0, f(x_0, y_0))$.

   (a) $f(x, y) = x^2 + 2xy + y^2$

   (b) $f(x, y) = e^{xy}$.

   (c) $f(x, y) = \sin x$.

   (a) We have $f_x = 2x + 2y$ and $f_y = 2x + 2y$, so plugging these into the tangent plane equation
   $$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$
   we get
   $$z - x_0^2 - 2x_0y_0 - y_0^2 = (2x_0 + 2y_0)(x - x_0) + (2x_0 + 2y_0)(y - y_0).$$

   (b) We have $f_x = ye^{xy}$ and $f_y = xe^{xy}$, so plugging this into the tangent plane equation (above), we get
   $$z - e^{x_0y_0} = y_0e^{x_0y_0}(x - x_0) + x_0e^{x_0y_0}(x - x_0).$$

   (c) We have $f_x = \cos x$ and $f_y = 0$, so the tangent plane equation is
   $$z - \sin x_0 = (\cos x_0)(x - x_0).$$

4. (a) Find $\frac{\mathrm{dz}}{\mathrm{d}t}$ for $z = xy^3 - x^2y$, $x = t^2 + 1$, $y = t^2 - 1$.

   (b) Find $\frac{\mathrm{d}w}{\mathrm{d}t}$ if $w = f(x, y, z) = xe^{y/z}$, if $x = t^2$, $y = 1 - t$, and $z = 1 + 2t$.

   (c) $w = xy + yz + zx$, $x = r\cos\theta$, $y = r\sin\theta$, $z = r\theta$, find the derivatives $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial \theta}$ when $r = 2$ and $\theta = \frac{\pi}{2}$

   (a) Using the chain rule, we have that $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$. Plugging these values in, we get:
   $$\frac{dz}{dt} = (y^3 - 2xy)(2t) + (3xy^2 - x^2)(2t)$$
   $$= \left((t^2 - 1)^3 - 2(t^2 + 1)(t^2 - 1)\right) * 2t + \left(3(t^2 + 1)(t^2 - 1)^2\right) - (t^2 + 1)^2\right)$$

   (b) If we change t, we will then change x, y, z, which then change w in turn. Thus by the chain rule, we have:
   $$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$
   Finally, evaluating this by plugging in partial derivatives, we have:
   $$\frac{dw}{dt} = (e^{y/z})(2t) + (\frac{x}{z}e^{y/z})(-1) - \frac{xy}{z^2}(2) = 2te^{y/z} - \frac{x}{z}e^{y/z} - \frac{2xy}{z^2}e^{y/z}$$
   Technically, you should replace all the x, y, and z with t, but that would make this very ugly, so I did not.

(c) By the same logic as part b, we have:

$$\frac{dw}{dr} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$
$$\frac{dw}{d\theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \theta}$$

Thus when we plug in the partial derivatives, we get:

$$\frac{dw}{dr} = (y+z)\cos\theta + (x+z)\sin\theta + (x+y)\theta$$

In order to get the numerical answer, we first need to solve for x, y, z at the given values of r, θ:

$$x = 2\cos(\frac{\pi}{2}) = 0,\ y = 2\sin(\frac{\pi}{2}) = 2,\ z = 2\frac{\pi}{2} = \pi$$

Thus we have $\frac{dw}{dr} = (2+\pi)\cos\frac{\pi}{2} + (0+\pi)\sin\frac{\pi}{2} + (0+2)\frac{\pi}{2} = \pi + \pi = 2\pi$.

Next, repeating the same thing for $\frac{dw}{d\theta}$:

$$\frac{dw}{d\theta} = (y+z)(-r\sin\theta) + (x+z)(r\cos\theta) + (x+y)r$$
$$\frac{dw}{d\theta} = (2+\pi)(-2\sin\frac{\pi}{2}) + (0+\pi)(2\cos\frac{\pi}{2}) + (2+0)2$$
$$\frac{dw}{d\theta} = -4 - 2\pi + 0 + 4 = -2\pi$$

5. (a) Use implicit differentiation to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for $y\cos x = x^2 + y^2$

   (b) Use implicit differentiation to find $\frac{\partial z}{\partial y}$ and $\frac{\partial z}{\partial x}$ for the equation $e^z = xyz$.

(a) First, we need to make this an equation equal to 0 like so:

$$y\cos x - x^2 - y^2 = 0$$

We can then use the formula like so:

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-y\sin x - 2x}{\cos x - 2y}$$

(b) We need to set the left hand side equal to 0 in order to apply implicit differentiation. We then have:

$$F(x,y,z) = e^z - xyz = 0$$

Finally, we know the formulas:

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} \text{ and } \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

Plugging the derivatives into these formulas, we have:

$$\frac{\partial z}{\partial y} = -\frac{-xz}{e^z - xy} \text{ and } \frac{\partial z}{\partial x} = -\frac{-yz}{e^z - xy}$$

Problems 1, 2, 4, and 5 courtesy of Theo Keller. Problem 3 courtesy of Carlos Esparza.