Fall 2025 **Math 53, Sections 104/106**

Discussion #2/3

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1. Write equations in polar coordinates to describe the following curves. Make sure to include the range for θ.

   (a) The curve $xy = 1$ for $x > 0$.
   Substitute in $x = r\cos(\theta), y = r\sin(\theta)$ to deduce that
   $$r^2\cos(\theta)\sin(\theta) = 1.$$
   Solving for r,
   $$r = \sqrt{\frac{1}{\cos(\theta)\sin(\theta)}}.$$
   The curve is the quadrant $x > 0, y > 0$ so we choose the range $0 < \theta < \pi/2$.

   (b) The parabola $x = y^2$.
   Substitute in $x = r\cos(\theta), y = r\sin(\theta)$ to deduce that
   $$r\cos(\theta) = r^2\sin^2(\theta).$$
   Solving for r,
   $$r = \frac{\cos(\theta)}{\sin^2(\theta)}.$$
   We can choose the range $0 < \theta < \pi$ to obtain the parabola.

   (c) The line $x = 1$.
   Substitute in $x = r\cos(\theta)$ to deduce that
   $$r\cos(\theta) = 1.$$
   Solving for r,
   $$r = \frac{1}{\cos(\theta)}.$$
   We choose the range $-\pi/2 < \theta < \pi/2$.

2. Consider the polar curve $r = 2\cos\theta$ for $0 \leqslant \theta \leqslant 2\pi$. Verify that it describes a circle centered at $(1, 0)$ with radius 1. How many times does it wrap around?

   We have $x = \cos(2\theta) + 1$ and $y = \sin(2\theta)$. The distance from $(1, 0)$ is $\sqrt{\cos^2(2\theta) + \sin^2(2\theta)} = 1$. It wraps around twice: it achieves distance two when $|\cos\theta| = 1$, i.e. $\theta = \pm\pi$.

3. (a) Find the area enclosed by the curve $x = t^2 - 2t, y = \sqrt{t}$ and the y-axis.

(b) Find the area enclosed by the x-axis and the curve $x = t^3 + 1, y = 2t - t^2$.

(a) The curve touches the y-axis exactly when $x = 0$, i.e. when $t = 0, 2$. Notice that we can write the curve as the graph of x as a function of y, i.e. $f(y) = x$. Eliminating the parameter, we obtain $t = y^2$, $f(y) = x = y^4 - 2y^2$. Since $x \leqslant 0$ for all $t \in [0, 2]$, the area enclosed is exactly the area under the curve of the graph of f , i.e. the integral of f. The interval of integration is $y \in \left[0, \sqrt{2}\right]$, so the area is

$$-\int_0^{\sqrt{2}} f(y)\,\mathrm{d}y = \frac{8}{15}\sqrt{2}.$$

Note that we had to add the sign to make the area make sense (be positive).

(b) Notice that $y = 2t - t^2$ intersects that x-axis at $t = 0$ and $t = 2$. The corresponding values of x are 1 and 9 so the area in question is

$$\int_1^9 y dx = \int_0^2 (2t - t^2)(3t^2) dt = 3\left[\frac{t^4}{2} - \frac{t^5}{5}\right]_0^2 = \frac{24}{5}.$$

4. Find the area of the region that lies inside $r = 3\cos\theta$ and outside $r = 1 + \cos\theta$.

First we find where these two curves intersect. Notice that if $3\cos\theta = 1 + \cos\theta \Leftrightarrow \cos\theta = 1/2$ so $\theta = -\pi/3, \pi/3$. Then by symmetry, the area is

$$\begin{aligned} 2\int_0^{\pi/3} \frac{1}{2}[(3\cos\theta)^2 - (1+\cos\theta)^2] d\theta &= \int_0^{\pi/3} 8\cos^2\theta - 2\cos\theta - 1 \\ &= \int_0^{\pi/3} 3 + 4\cos 2\theta - 2\cos\theta d\theta \\ &= \pi. \end{aligned}$$

5. Find the length of each curve:

(a) $r = 2\cos\theta, 0 \leq \theta \leq \pi$.

(b) $r = \theta^2, 0 \leq \theta \leq 2\pi$.

(a) Applying the polar formula we have

$$L = \int_0^{\pi} \sqrt{r^2 + (dr/d\theta)^2} d\theta = \int_0^{\pi} \sqrt{4\cos^2\theta + 4\sin^2\theta} d\theta = 2\pi.$$

(b) Applying the polar formula we have

$$L = \int_0^{2\pi} \sqrt{\theta^4 + 4\theta^2} d\theta = \int_0^{2\pi} \theta\sqrt{\theta^2 + 4} d\theta = \frac{1}{3}(\theta^2 + 4)^{3/2}\Big|_0^{2\pi} = \frac{8}{3}[(\pi^2 + 1)^{3/2} - 1].$$

6. (a) If $\vec{u}$ and $\vec{v}$ are unit vectors in $\mathbb{R}^3$ and $u \circ v = -1$, what is the angle between $\vec{u}$ and $\vec{v}$?

From the formula $\vec{u} \circ \vec{v} = |\vec{u}| \cdot |\vec{v}| \cos\theta$, it follows that $\cos\theta = -1$, so $\theta = \pi$.

(b) Find three nonzero vectors in $\mathbb{R}^3$ that are perpendicular to $\langle 1, 3, 2\rangle$.

A nonzero vector $\langle x, y, z\rangle$ will work if and only if $x + 3y + 2z = 0$. Specifically $\langle -1, 1, -1\rangle$ and $\langle 2, 0, -1\rangle$, and $\langle 3, -1, 0\rangle$ all work (alternatively, once one solution is found, it may be scaled to find others).

(c) Let P be a vertex on a cube. Let Q be an adjacent vertex and let R be the vertex opposite to P. Using dot products, find the angle between the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

Without loss of generality, take the cubic to lie in the first octant, with edges along the positive coordinate axes, and have edges of length 1, so that $P = (0, 0, 0)$ and $Q = (1, 0, 0)$. Then $R = (1, 1, 1)$ and $\overrightarrow{PQ} = \langle 1, 0, 0\rangle$. Similarly $\overrightarrow{PR} = \langle 1, 1, 1\rangle$, so $\overrightarrow{PQ} \circ \overrightarrow{PR} = 1 = |\overrightarrow{PQ}| \cdot |\overrightarrow{PR}| \cos(\theta) = \sqrt{3}$, so $\theta = \arccos(1/\sqrt{3})$

(d) If $\vec{u}$ and $\vec{v}$ are unit vectors in $\mathbb{R}^3$, show that the vectors $\vec{u} + \vec{v}$ and $\vec{v} - \vec{v}$ are perpendicular.

We have $(\vec{u} + \vec{v}) \circ (\vec{u} - \vec{v}) = |\vec{u}|^2 - |\vec{v}|^2 = 1 - 1 = 0$.

(e) Find the vector projection of $\vec{v}$ onto $\vec{w}$ and the scalar projection of $\vec{v}$ onto $\vec{w}$ if $\vec{v} = \langle 2, 4\rangle$, $\vec{w} = \langle 3, 1\rangle$.

The vector projection is

$$\frac{\vec{v} \cdot \vec{w}}{|\vec{w}|^2}\vec{w} = \frac{2 \cdot 3 + 4 \cdot 1}{3^2 + 1^2}\langle 3, 1\rangle = \frac{10}{10}\langle 3, 1\rangle = \langle 3, 1\rangle,$$

and the scalar projection is

$$\frac{\vec{v} \cdot \vec{w}}{|\vec{w}|} = \frac{2 \cdot 3 + 4 \cdot 1}{\sqrt{3^2 + 1^2}} = \frac{10}{\sqrt{10}} = \sqrt{10}.$$

7. (a) Find the cross products $\vec{v} \times \vec{w}$ if $\vec{v} = \langle 2, 3, 1\rangle$ and $\vec{w} = \langle -1, 2, 3\rangle$.

We use the determinant formula:

$$\begin{aligned}
\vec{v} \times \vec{w} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 1 \\ -1 & 2 & 3 \end{vmatrix} \\
&= \begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix}\vec{i} - \begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix}\vec{j} + \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix}\vec{k} \\
&= (3^2 - 1 \cdot 2)\vec{i} - (2 \cdot 3 - 1 \cdot (-1))\vec{j} + (2 \cdot 2 - 3 \cdot (-1))\vec{k} \\
&= 7\vec{i} - 8\vec{j} + 7\vec{k}.
\end{aligned}$$

(b) Let $\vec{u}$ and $\vec{v}$ be nonzero vectors with $\vec{u} \times \vec{v} = \vec{0}$. What can you say about the relationship between $\vec{u}$ and $\vec{v}$?

Let θ be the angle between the two vectors; then we have

$$0 = |\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta.$$

This can only happen if $\sin\theta = 0$, which implies that θ is an integer multiple of π. Thus we may conclude that $\vec{u}$ and $\vec{v}$ are collinear.

(c) Find the area of the triangle with two sides given by the vectors $\vec{v} = \langle 1, 2 \rangle$ and $\vec{w} = \langle -3, 4 \rangle$.

We view this triangle as sitting within the xy-plane in $\mathbb{R}^3$. Then the quantity $|\vec{v} \times \vec{w}|$ gives the area of the parallelogram with two sides given by $\vec{v}$ and $\vec{w}$. We compute

$$\vec{v} \times \vec{w} = \langle 0, 0, 1 \cdot 4 - 2 \cdot (-3) \rangle = 10\vec{k},$$

where we are justified in ignoring the $\vec{i}$ and $\vec{j}$ components because we know that $\vec{v} \times \vec{w}$ must be orthogonal to the xy-plane. So the area of this parallelogram is 10. The area of the triangle is half that of the parallelogram, so we see that the desired area is 5.

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