Fall 2025 **Math 53, Sections 104/106**

Discussion #32/33

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1. Evaluate the surface integral $\iint_S \vec{F} \cdot d\vec{S}$ for the given vector field $\vec{F}$ and oriented surface S. For closed surfaces, use the positive (outward) orientation.

   (a) $\vec{F}(x,y,z) = \langle ze^{xy}, -3ze^{xy}, xy\rangle$. S is the parallelogram $x = u+v, y = u-v, z = 1+2u+v, 0 \leqslant u \leqslant 2, 0 \leqslant v \leqslant 1$ oriented upwards.
   Let $\vec{r}(u,v) = \langle u+v, u-v, 1+2u+v\rangle$. Then $\vec{r}_u \times \vec{r}_v = \langle 3, 1, -2\rangle$. We get $\vec{F}(\vec{r}(u,v)) = \langle (1+2u+v)e^{u^2-v^2}, -3(1+2u+v)e^{u^2-v^2}, u^2-v^2\rangle$. Since the z-component of $\vec{r}_u \times \vec{r}_v$ is negative, we use $-(\vec{r}_u \times \vec{r}_v)$. Thus,
   $$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (-(\vec{r}_u \times \vec{r}_v))\, dA = \int_0^1 \int_0^2 2(u^2 - v^2)\, du\, dv = 4.$$

   (b) $\vec{F}(x,y,z) = \langle 0, y, -z\rangle$ and S consists of the paraboloid $y = x^2 + z^2, 0 \leqslant y \leqslant 1$, and the disk $x^2 + z^2 \leqslant 1, y = 1$.
   Let S_1 be the paraboloid and S_2 be the disk. Since S is closed, we use the outward orientation. On S_1 we have $\vec{F}(\vec{r}(x,z)) = \langle 0, x^2+z^2, -z\rangle$ and $\vec{r}_x \times \vec{r}_z = \langle 1, 2x, 0\rangle \times \langle 0, 2z, 1\rangle = \langle 2x, -1, 2z\rangle$. Then
   $$\iint_{S_1} \vec{F} \cdot d\vec{S} = \iint_{x^2+z^2 \leqslant 1} (-(x^2+z^2) - 2z^2)\, dA = -\int_0^{2\pi} \int_0^1 (r^2 + 2r^2 \sin^2\theta) r\, dr\, d\theta = -\pi$$
   and on S_2 we have $\vec{F}(\vec{r}(x,z)) = \langle 0, 1, -z\rangle$ and $\vec{r}_z \times \vec{r}_x = \langle 0, 1, 0\rangle$. Then $\iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_{x^2+z^2 \leqslant 1} dA = \pi$.
   This can also be done with the divergence theorem.

   (c) $\vec{F}(x,y,z) = \langle x^2, y^2, z^2\rangle$ and S is the boundary of the solid half cylinder $0 \leqslant z \leqslant \sqrt{1-y^2}, 0 \leqslant x \leqslant 2$.
   Here S has four surfaces. S_1 is the portion of the cylinder, S_2 is the bottom surfaces (lies on xy-plane), S_3 is the front half disk at $x = 2$ and S_4 is the back half disk at $x = 0$. On S_1 we have $\vec{r}(x,y) = \langle x, y, \sqrt{1-y^2}\rangle$ so $\vec{r}_x = \langle 1, 0, 0\rangle$ and $\vec{r}_y = \langle 0, 1, -y(1-y^2)^{-1/2}$. Then
   $$\iint_{S_1} F \cdot d\vec{S} = \int_0^2 \int_{-1}^1 \vec{F}(\vec{r}(x,y)) \cdot (\vec{r}_x \times \vec{r}_y)\, dx\, dy = \int_0^2 \int_{-1}^1 y^3(1-y^2)^{-1/2} + (1-y^2)\, dy\, dx = 8/3.$$
   On S_2 we have $z = 0$ with downward orientation so $\iint_{S_2} \vec{F} \cdot d\vec{S} = \int_0^2 \int_{-1}^1 -z^2\, dy\, dx = 0$. On S_3, the surfaces is $x = 2$ for $-1 \leqslant y \leqslant 1$ and $0 \leqslant z \leqslant \sqrt{1-y^2}$ oriented in the positive x-direction. Hence, $\vec{r}_y \times r_z = \vec{i}$ so $\iint_{S_3} \vec{F} \cdot d\vec{S} = \int_{-1}^1 \int_0^{\sqrt{1-y^2}} x^2\, dz\, dy =$

$4\int_{-1}^{1}\int_0^{\sqrt{1-y^2}} \mathrm{d}z\,\mathrm{d}y = 2\pi$. On S_4, the surfaces is $x = 0$ for $-1 \leqslant y \leqslant 1$ and $0 \leqslant z \leqslant \sqrt{1-y^2}$ oriented in the negative x-direction. Hence, $\vec{r}_z \times r_y = -\vec{i}$ so $\iint_{S_3} \vec{F}\cdot \mathrm{d}\vec{S} = \int_{-1}^{1}\int_0^{\sqrt{1-y^2}} x^2\,\mathrm{d}z\,\mathrm{d}y = 0\int_{-1}^{1}\int_0^{\sqrt{1-y^2}} \mathrm{d}z\,\mathrm{d}y = 0$. Summing these we get $2\pi + 8/3$.

This can also be done using the divergence theorem.

2. Let S be the cylinder $x^2+y^2 = 1$, $-1 \leqslant z \leqslant 1$, plus its top and bottom caps. Compute the flux of the vector field

$$\vec{F}(x,y,z) = \begin{pmatrix} -\sin \pi y \\ -\cos \pi x \\ xy \end{pmatrix}$$

both directly and by using the divergence theorem.

The vector field is incompressible, so by the divergence theorem we immediately know that the flux has to be zero. If we do a direct computation the sides of the cylinder will have zero contribution because $\vec{n} \perp \vec{F}$ there and the contribution from the top and bottom caps will cancel.

3. (a) Compute $\iint_S \vec{F}\cdot \mathrm{d}\vec{S}$ where $\vec{F} = (x^2, 2z, -3y)$ and S is the portion of $y^2+z^2 = 4$ between $x = 0$ and $x = 3-z$.
   (b) Compute $\iint_S (\nabla \times \vec{F})\cdot \mathrm{d}\vec{S}$ where $\vec{F} = (y, -x, yx^3)$ and S is the portion of the sphere of radius 4 with $z \geq 0$ and the upwards orientation.
   (c) Compute $\iint_S \vec{F}\cdot \mathrm{d}\vec{S}$ where $\vec{F} = (\sin(\pi x), zy^3, z^2+4x)$ where S is the surface of the box $-1 \leqslant x \leqslant 2$, $0 \leqslant y \leqslant 1$, and $1 \leqslant z \leqslant 4$, oriented outwards.

(a) Parametrize the surface by $x = x$, $y = 2\cos\theta$, and $z = 2\sin\theta$ for $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant x \leqslant 3 - 2\cos\theta$. Then $\vec{r}_x = (1,0,0)$ and $\vec{r}_\theta = (0, 2\cos\theta, -2\sin\theta)$. So $\vec{r}_x \times \vec{r}_\theta = -2\sin\theta\vec{j} - 2\cos\theta\vec{k}$. Our integral then becomes

$$\begin{aligned}\int_S \vec{F}\cdot \mathrm{d}\vec{S} &= \int_0^{2\pi}\int_0^{3-2\cos\theta} (0, 2\cos\theta, -2\sin\theta)\cdot(x^2, 4\cos\theta, -6\sin\theta)\,\mathrm{d}x\,\mathrm{d}\theta \\ &= \int_0^{2\pi}\int_0^{3-2\cos\theta} 4\sin\theta\cos\theta\,\mathrm{d}x\,\mathrm{d}\theta \\ &= \int_0^{2\pi} 12\sin\theta\cos\theta - 8\sin\theta\cos^2\theta\,\mathrm{d}\theta = 0.\end{aligned}$$

(b) We use Stokes' theorem and then Green's theorem. Note that the boundary circle C is the circle of radius 4 centered at the origin in the xy-plane. Let D be the disk of radius 4 enclosed by C in the xy-plane. Then

$$\begin{aligned}\iint_S (\nabla\times\vec{F})\cdot \mathrm{d}\vec{S} &= \int_C \vec{F}\cdot d\vec{r} \\ &= \int_C y\,\mathrm{d}x - x\,\mathrm{d}y \\ &= \iint_D -2\,\mathrm{d}A = -2\cdot(16\pi) = -32\pi.\end{aligned}$$

(c) We use the divergence theorem. Note that $\nabla \cdot \vec{F} = \pi\cos(\pi x) + 3y^2 z + 2z$. So our integral becomes

$$\begin{aligned}\iint_S \vec{F} \cdot \mathrm{d}\vec{S} &= \int_{-1}^{2}\int_{0}^{1}\int_{1}^{4} (\pi\cos(\pi x) + 3y^2 z + 2z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{-1}^{2}\int_{0}^{1} 3\pi\cos(\pi x) + \frac{45}{2}y^2 + 15\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{-1}^{2} 3\pi\cos(\pi x) + \frac{15}{2} + 15\,\mathrm{d}x = \frac{135}{2}.\end{aligned}$$

4. Let $\vec{F}(x,y,z) = (x^2, yz, xz)$ and evaluate $\iint_S \nabla \times \vec{F} \cdot \mathrm{d}\vec{S}$, where S is the unit sphere...

- by direct computation.
- using a symmetry argument.
- using the divergence theorem.
- using Stokes' theorem.

All problems courtesy of Carlos Esparza.