

Discussion #29

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1. Consider a sphere of radius R centered at the origin. We know that the sphere can be parametrized by

$$\vec{r}(\phi, \theta) = \begin{pmatrix} R \sin \phi \cos \theta \\ R \sin \phi \sin \theta \\ R \cos \phi \end{pmatrix},$$

$$0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi.$$

- (a) Compute the partial derivatives of $\vec{r}(\phi, \theta)$.

$$\vec{r}_\phi = \langle R \cos \phi \cos \theta, R \cos \phi \sin \theta, -R \sin \phi \rangle \text{ and } \vec{r}_\theta = \langle -R \sin \phi \sin \theta, R \sin \phi \cos \theta, 0 \rangle$$

- (b) Compute the normal vector $\vec{r}_\phi \times \vec{r}_\theta$ produced by this parametrization. Express it in terms of ϕ, θ and x, y, z .

$$\begin{aligned} \vec{N}(\phi, \theta) &= \vec{r}_\phi \times \vec{r}_\theta \\ &= \langle R^2 \sin^2 \phi \cos \theta, R^2 \sin^2 \phi \sin \theta, R^2 \sin \phi \cos \phi \rangle \\ &= R \sin \phi \langle R \sin \phi \cos \theta, R \sin \phi \sin \theta, R \cos \phi \rangle \\ &= R \sin \phi \vec{r}(\theta, \phi) \\ &= R \sin \phi \cdot \langle x, y, z \rangle \end{aligned}$$

- (c) Use the magnitude of the normal vector (the "Jacobian") to compute the area of the unit sphere.

We can now obtain the area of the sphere by integrating $|\vec{N}| = R^2 \sin \phi$ over $0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi$:

$$A = \int_0^\pi \int_0^{2\pi} R^2 \sin \phi \, d\theta \, d\phi = 2\pi R^2 \int_0^\pi \sin \phi \, d\phi = 4\pi R^2$$

- (d) Compute the surface integral of z^2 over the sphere. This is

$$A = \int_0^\pi \int_0^{2\pi} R^2 \cos^2 \phi R^2 \sin \phi \, d\theta \, d\phi = 2\pi R^4 \int_0^\pi \cos^2 \phi \sin \phi \, d\phi = 4\pi R^4 / 3$$

2. Parametrize the following surfaces in an appropriate way (if they are not already parametrized) and compute their normal vectors and area.

- (a) The portion of the elliptic paraboloid $z = x^2 + y^2$ lying over the unit disk.

This surface is the graph of $f(x, y) = x^2 + y^2$, so we know that

$$\vec{N} = \langle 1, 0, f_x \rangle \times \langle 0, 1, f_y \rangle = \langle -f_x, -f_y, 1 \rangle = \langle -2x, -2y, 1 \rangle.$$

The area is computed by the following integral over the unit disk D , which we compute in polar coordinates and using the substitution $u = 1 + 4r^2$

$$\begin{aligned} \int_D |\vec{N}| \, dA &= \int_D \sqrt{1 + 4x^2 + 4y^2} \, dA = \int_0^{2\pi} \int_0^1 \sqrt{1 + 4r^2} \, r \, dr \, d\phi \\ &= 2\pi \int_1^5 \sqrt{u} \frac{1}{8} \, du = \frac{\pi}{6} (5^{3/2} - 1) \end{aligned}$$

- (b) The ellipsoid $2z^2 + x^2 + y^2 = 1$. You don't need to evaluate the integral, but you can do it using

$$\int \sqrt{1 + x^2} \, dx = \frac{1}{2} x \sqrt{1 + x^2} + \frac{1}{2} \ln(x + \sqrt{1 + x^2}) + C.$$

This ellipsoid can be obtained by taking the unit sphere and squishing the z -coordinate by a factor of $\frac{1}{2}$. Hence we can parametrize it by (cf problem ??)

$$\vec{r}(\phi, \theta) = \left\langle \sin \phi \cos \theta, \sin \phi \sin \theta, \frac{1}{2} \cos \phi \right\rangle$$

Now we obtain

$$\begin{aligned} \vec{r}_\phi &= \left\langle \cos \phi \cos \theta, \cos \phi \sin \theta, -\frac{1}{2} \sin \phi \right\rangle \\ \vec{r}_\theta &= \langle -\sin \phi \sin \theta, \sin \phi \cos \theta, 0 \rangle \\ \vec{N} = \vec{r}_\phi \times \vec{r}_\theta &= \left\langle \frac{1}{2} \sin^2 \phi \cos \theta, \frac{1}{2} \sin^2 \phi \sin \theta, \sin \phi \cos \phi \right\rangle \end{aligned}$$

and we can set up an integral for the area:

$$\begin{aligned} A &= \int_0^{2\pi} \int_0^\pi |\vec{N}| \, d\phi \, d\theta \\ &= \int_0^{2\pi} \int_0^\pi \sin \phi \sqrt{\frac{1}{4} \sin^2 \phi \cos^2 \theta + \frac{1}{4} \sin^2 \phi \sin^2 \theta + \cos^2 \phi} \, d\phi \, d\theta \\ &= 2\pi \int_0^\pi \sin \phi \sqrt{\frac{1}{4} \sin^2 \phi + \cos^2 \phi} \, d\phi = \pi \int_0^\pi \sin \phi \sqrt{1 + 3 \cos^2 \phi} \, d\phi \\ &= \pi \int_{-1}^1 \sqrt{1 + 3u^2} \, du = \pi \frac{1}{\sqrt{3}} \frac{1}{2} \left[x \sqrt{1 + x^2} + \ln(x + \sqrt{1 + x^2}) \right]_{x=-\sqrt{3}}^{x=\sqrt{3}} \\ &= \pi \left(2 + \frac{\ln(2 + \sqrt{3})}{\sqrt{3}} \right) \approx 8.672 \end{aligned}$$

(c) The parametric surface $\vec{r}(u, v) = (u^2, uv, v^2/2)$ where $0 \leq u \leq 1, 0 \leq v \leq 2$.

$\vec{r}_u = \langle 2u, v, 0 \rangle, \vec{r}_v = \langle 0, u, v \rangle$ so $\vec{N} = \vec{r}_u \times \vec{r}_v = \langle v^2, -2uv, 2u^2 \rangle$. The area of the surface is

$$\begin{aligned} A &= \iint_{[0,1] \times [0,2]} |\vec{N}| \, dA = \int_0^1 \int_0^2 \sqrt{v^4 + 4u^2v^2 + 4u^4} \, dv \, du = \int_0^1 \int_0^2 \sqrt{(v^2 + 2u^2)^2} \, dv \, du \\ &= \int_0^1 \int_0^2 v^2 + 2u^2 \, dv \, du = \frac{8}{3} + \frac{4}{3} = 4. \end{aligned}$$

(d) The part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$.

This is the graph of $f(x, y) = xy$, so

$$\vec{N} = \langle -f_x, -f_y, 1 \rangle = \langle y, x, 1 \rangle.$$

Restricting the surface to the part inside the cylinder corresponds to restricting the domain of f to the unit disk D . The area of the surface is given by

$$A = \iint_D \sqrt{1 + x^2 + y^2} \, dA = \int_0^{2\pi} \int_0^1 \sqrt{r^2 + 1} r \, dr \, d\theta = \frac{2\pi}{3} (2\sqrt{2} - 1)$$

3. Compute the surface integral

$$\iint_S f(x, y, z) \, dS$$

for the given function $f(x, y, z)$ over the surface S .

(a) $f(x, y, z) = x$ where S is the surface $y = x^2 + 4z, 0 \leq x \leq 1, 0 \leq z \leq 1$.

Using x and z as parameters we have $\vec{r}(x, z) = \langle x, x^2 + 4z, z \rangle, 0 \leq x \leq 1, 0 \leq z \leq 1$. Then

$$\begin{aligned} \vec{N}(x, z) &= \vec{r}_x \times \vec{r}_z = (\vec{i} + 2x\vec{j}) \times (4\vec{j} + \vec{k}) = \langle 2x, -1, 4 \rangle \\ \iint_S x \, dS &= \int_0^1 \int_0^1 x \cdot |\vec{N}(x, z)| \, dx \, dz = \int_0^1 x \sqrt{4x^2 + 17} \, dx \\ &= \left[\frac{1}{8} \cdot \frac{2}{3} (4x^2 + 17)^{3/2} \right]_0^1 = \frac{7}{4} \sqrt{21} - \frac{17}{12} \sqrt{17}. \end{aligned}$$

(b) $f(x, y, z) = (x^2 + y^2)z$ and S the hemisphere $x^2 + y^2 + z^2 = 4, z \geq 0$.

We parametrize the sphere as in problem ??, so $(x^2 + y^2)z = R^3 \sin^2 \phi \cos \phi = 8 \sin^2 \phi \cos \phi$ because $R = 2$. We compute

$$\begin{aligned} \iint_S f(x, y, z) \, dS &= \int_0^{\pi/2} \int_0^{2\pi} 8 \sin^2 \phi \cos \phi \cdot 4 \sin \phi \, d\theta \, d\phi \\ &= 64\pi \int_0^{\pi/2} \sin^3 \phi \cos \phi \, d\phi = 64\pi \int_0^1 u^3 \, du = 16\pi \end{aligned}$$

All problems courtesy of Carlos Esparza.