Fall 2025 **Math 53, Sections 104/106**

Discussion #29

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1. Consider a sphere of radius R centered at the origin. We know that the sphere can be parametrized by

$$\vec{r}(\phi,\theta) = \begin{pmatrix} R\sin\phi\cos\theta \\ R\sin\phi\sin\theta \\ R\cos\phi \end{pmatrix},$$

$0 \leqslant \phi \leqslant \pi, 0 \leqslant \theta \leqslant 2\pi$.

   (a) Compute the partial derivatives of $\vec{r}(\phi,\theta)$.

   (b) Compute the normal vector $\vec{r}_u \times \vec{r}_v$ produced by this parametrization. Express it in terms of ϕ, θ and x, y, z.

   (c) Use the magnitude of the normal vector (the "Jacobian") to compute the area of the unit sphere.

   (d) Compute the surface integral of z^2 over the sphere.

2. Parametrize the following surfaces in an appropriate way (if they are not already parametrized) and compute their normal vectors and area.

   (a) The portion of the elliptic paraboloid $z = x^2 + y^2$ lying over the unit disk.

   (b) The ellipsoid $2z^2 + x^2 + y^2 = 1$. You don't need to evaluate the integral, but you can do it using

$$\int \sqrt{1+x^2}\,\mathrm{d}x = \frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\ln\left(x + \sqrt{1+x^2}\right) + C.$$

   (c) The parametric surface $\vec{r}(u,v) = (u^2, uv, v^2/2)$ where $0 \leqslant u \leqslant 1, 0 \leqslant v \leqslant 2$.

   (d) The part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$.

3. Compute the surface integral

$$\iint_S f(x,y,z)\,\mathrm{d}S$$

for the given function $f(x,y,z)$ over the surface S.

(a) $f(x,y,z) = x$ where S is the surface $y = x^2 + 4z, 0 \leqslant x \leqslant 1, 0 \leqslant z \leqslant 1$.

(b) $f(x,y,z) = (x^2 + y^2)z$ and S the hemisphere $x^2 + y^2 + z^2 = 4, z \geq 0$.