Fall 2025 **Math 53, Sections 104/106**

Discussion #27

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1. If $\nabla \times \vec{F} = 0$, show that $\vec{F}$ is conservative.

   $\vec{F}$ has components $\langle P, Q, R\rangle$ and $0 = \nabla \times F = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle$, so taking each of the three scalar equations from the vector equations and differentiating in x, y, and z, respectively, and using Clairaut's theorem, recovers the conservative vector condition.

2. For each of the following vector fields $\vec{F}$, compute its curl and divergence. State whether each vector field is irrotational, incompressible, or neither.

   (a) $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$

   We have
   $$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z = 1 + 1 + 1 = 3$$
   and
   $$\nabla \times \vec{F} = \left(\frac{\partial}{\partial y}z - \frac{\partial}{\partial z}y\right)\vec{i} + \left(\frac{\partial}{\partial z}x - \frac{\partial}{\partial x}z\right)\vec{j} + \left(\frac{\partial}{\partial x}y - \frac{\partial}{\partial y}x\right)\vec{k} = \vec{0}.$$
   Because $\nabla \times \vec{F} = \vec{0}$, we see that $\vec{F}$ is irrotational.

   (b) $\vec{F} = \langle y^2, z^3, x^4\rangle$

   We have
   $$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}y^2 + \frac{\partial}{\partial y}z^3 + \frac{\partial}{\partial z}x^3 = 0 + 0 + 0 = 0$$
   and
   $$\begin{aligned}\nabla \times \vec{F} &= \left(\frac{\partial}{\partial y}x^4 - \frac{\partial}{\partial z}z^3\right)\vec{i} + \left(\frac{\partial}{\partial z}y^2 - \frac{\partial}{\partial x}x^4\right)\vec{j} + \left(\frac{\partial}{\partial x}z^3 - \frac{\partial}{\partial y}y^2\right)\vec{k} \\ &= -3z^2\vec{i} - 4x^3\vec{j} - 2y\vec{k}.\end{aligned}$$
   Because $\nabla \cdot \vec{F} = 0$, we see that $\vec{F}$ is incompressible.

   (c) $\vec{F} = \langle y^2x, e^z, z^2\rangle$

   We have
   $$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}y^2x + \frac{\partial}{\partial y}e^z + \frac{\partial}{\partial z}z^2 = y^2 + 2z$$
   and
   $$\begin{aligned}\nabla \times \vec{F} &= \left(\frac{\partial}{\partial y}z^2 - \frac{\partial}{\partial z}e^z\right)\vec{i} + \left(\frac{\partial}{\partial z}y^2x - \frac{\partial}{\partial x}z^2\right)\vec{j} + \left(\frac{\partial}{\partial x}e^z - \frac{\partial}{\partial y}y^2x\right)\vec{k} \\ &= -e^z\vec{i} - 2yx\vec{k}.\end{aligned}$$
   This vector field is neither irrotational nor incompressible (as $\nabla \cdot \vec{F}$ and $\nabla \times \vec{F}$ are both nonzero).

(d) $\vec{F} = \nabla f$, where $f(x,y,z) = 2xye^{yz}$

We have

$$\nabla \cdot \vec{F} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 + (4xze^{yz} + 2xyz^2 e^{yz}) + 2xy^3 e^{yz} = (4xz + 2xyz^2 + 2xy^3)e^{yz}$$

and

$$\nabla \times \vec{F} = \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\vec{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial y}\right)\vec{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\vec{k} = \vec{0}$$

by Clairaut's theorem.

3. Let $\vec{F} = \langle P, Q, R\rangle$ and $\vec{G} = \langle P', Q', R'\rangle$ be vector fields on $\mathbb{R}^3$, and let $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}$ be functions on $\mathbb{R}^3$. Assume all of these are infinitely differentiable. Prove each of the following vector identities.

(a) $\nabla \cdot (f\vec{F}) = f(\nabla \cdot \vec{F}) + \vec{F} \cdot (\nabla f)$

$$\begin{aligned}
\nabla \cdot (f\vec{F}) &= (fP)_x + (fQ)_y + (fR)_z \\
&= (f_xP + fP_x) + (f_yQ + fQ_y) + (f_zR + fR_z) \\
&= (fP_x + fQ_y + fR_z) + (f_xP + f_yQ + f_zR) \\
&= f(\nabla \cdot \vec{F}) + \vec{F} \cdot (\nabla f).
\end{aligned}$$

(b) $\nabla \times (f\vec{F}) = f(\nabla \times \vec{F}) + (\nabla f) \times \vec{F}$

It's easiest to verify this component-by component. For the $\vec{i}$ component, we have

$$\begin{aligned}
(\nabla \times (f\vec{F})) \cdot \vec{i} &= (fR)_y - (fQ)_z \\
&= f_yR + fR_y - f_zQ - fQ_z \\
&= f(R_y - Q_z) + (f_yR - f_zQ) \\
&= f(\nabla \times \vec{F}) \cdot \vec{i} + ((\nabla f) \times \vec{F}) \cdot \vec{i} \\
&= (f(\nabla \times \vec{F}) + (\nabla f) \times \vec{F}) \cdot \vec{i}.
\end{aligned}$$

A similar computation shows that the identity holds for the $\vec{j}$ and $\vec{k}$ components, proving the the identity.

(c) $\nabla \cdot (\vec{F} \times \vec{G}) = \vec{G} \cdot (\nabla \times \vec{F}) - \vec{F} \cdot (\nabla \times \vec{G})$

$$\begin{aligned}
\nabla \cdot (\vec{F} \times \vec{G}) &= (QR' - RQ')_x + (RP' - PR')_y + (PQ' - QP')_z \\
&= Q_xR' + QR'_x - R_xQ' - RQ'_x + R_yP' + RP'_y - P_yR' - PR'_y + P_zQ' + PQ'_z - Q_zP' - QP'_z \\
&= P'(R_y - Q_z) + Q'(P_z - R_x) + R'(Q_x - P_y) - P(R'_y - Q'_z) - Q(P'_z - R'_x) - R(Q'_x - P'_y) \\
&= \vec{G} \cdot (\nabla \times \vec{F}) - \vec{F} \cdot (\nabla \times \vec{G}).
\end{aligned}$$

(d) $\nabla \cdot (\nabla f \times \nabla g) = 0$

This follows from the previous identity (taking $\vec{F} = \nabla f$ and $\vec{G} = \nabla g$) and the fact that $\nabla \times (\nabla f) = \nabla \times (\nabla g) = 0$.

(e) $\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$

We again work component-by-component. Note

$$\nabla \times \vec{F} = (R_y - Q_z)\vec{i} + (P_z - R_x)\vec{j} + (Q_x - P_y)\vec{k},$$

so

$$\begin{aligned}
(\nabla \times (\nabla \times \vec{F})) \cdot \vec{i} &= (Q_x - P_y)_y - (P_z - R_x)_z \\
&= Q_{xy} - P_{yy} - P_{zz} + R_{xz} \\
&= (P_{xx} + Q_{xy} + R_{xz}) - (P_{xx} + P_{yy} + P_{zz}) \\
&= (P_x + Q_y + R_z)_x - (P_{xx} + P_{yy} + P_{zz}) \\
&= (\nabla(\nabla \cdot \vec{F})) \cdot \vec{i} - (\nabla^2 \vec{F}) \cdot \vec{i} \\
&= \left(\nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}\right) \cdot \vec{i}.
\end{aligned}$$

Similar arguments show the identity also holds in the $\vec{j}$ and $\vec{k}$ components, proving the identity.

4. **(Challenge)** Suppose you are given a pair of (infinitely differentiable) vector fields $\vec{E}$ and $\vec{B}$ in $\mathbb{R}^3$ in $\mathbb{R}^3$, and consider each vector field as additionally varying with respect to a variable t (in addition to the variables x, y, and z for $\mathbb{R}^3$). Suppose furthermore that these vector fields satisfy the "Maxwell equations in a vacuum:"

$$\nabla \cdot \vec{E} = 0 \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

for some constant $c^2 > 0$. Prove that these vector fields satisfy the "wave equations"

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} \qquad \nabla^2 \vec{B} = \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2}.$$

Here $\nabla^2 \vec{E}$ is the vector Laplacian

$$\nabla^2 \vec{E} = \frac{\partial^2 \vec{E}}{\partial x^2} + \frac{\partial^2 \vec{E}}{\partial y^2} + \frac{\partial^2 \vec{E}}{\partial z^2},$$

and $\nabla^2 \vec{B}$ is defined similarly (with $\vec{E}$ replaced by $\vec{B}$).

By completing this exercise, you are showing that the fundamental laws of electrodynamics suggest the possibility of electromagnetic waves, i.e. light. *Fiat lux!*

The last problem from the above shows that

$$\nabla^2 \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla \times (\nabla \times \vec{E}).$$

Substituting in the equations for $\nabla \cdot \vec{E}$ and $\nabla \times \vec{E}$ here gives

$$\nabla^2 \vec{E} = 0 - \nabla \times \left(-\frac{\partial \vec{B}}{\partial t} \right) = \nabla \times \frac{\partial \vec{B}}{\partial t}.$$

By Clairaut's theorem, we can replace this with

$$\nabla^2 \vec{E} = \frac{\partial}{\partial t} \left(\nabla \times \vec{B} \right).$$

Substituting in the equation for $\nabla \times \vec{B}$ gives

$$\nabla^2 \vec{E} = \frac{\partial}{\partial t} \left(\frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \right) = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}.$$

The proof of the wave equation for $\vec{B}$ is similar.

Problems 2–4 courtesy of Carlos Esparza.