Fall 2025 **Math 53, Sections 104/106**

Discussion #27

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1. If $\nabla \times \vec{F} = 0$, show that $\vec{F}$ is conservative.

2. For each of the following vector fields $\vec{F}$, compute its curl and divergence. State whether each vector field is irrotational, incompressible, or neither.

   (a) $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$
   (b) $\vec{F} = \langle y^2, z^3, x^4 \rangle$
   (c) $\vec{F} = \langle y^2 x, e^z, z^2 \rangle$
   (d) $\vec{F} = \nabla f$, where $f(x, y, z) = 2xye^{yz}$

3. Let $\vec{F} = \langle P, Q, R \rangle$ and $\vec{G} = \langle P', Q', R' \rangle$ be vector fields on $\mathbb{R}^3$, and let $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}$ be functions on $\mathbb{R}^3$. Assume all of these are infinitely differentiable. Prove each of the following vector identities.

   (a) $\nabla \cdot (f\vec{F}) = f(\nabla \cdot \vec{F}) + \vec{F} \cdot (\nabla f)$
   (b) $\nabla \times (f\vec{F}) = f(\nabla \times \vec{F}) + (\nabla f) \times \vec{F}$
   (c) $\nabla \cdot (\vec{F} \times \vec{G}) = \vec{G} \cdot (\nabla \times \vec{F}) - \vec{F} \cdot (\nabla \times \vec{G})$
   (d) $\nabla \cdot (\nabla f \times \nabla g) = 0$
   (e) $\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$

4. **(Challenge)** Suppose you are given a pair of (infinitely differentiable) vector fields $\vec{E}$ and $\vec{B}$ in $\mathbb{R}^3$ in $\mathbb{R}^3$, and consider each vector field as additionally varying with respect to a variable t (in addition to the variables x, y, and z for $\mathbb{R}^3$). Suppose furthermore that these vector fields satisfy the "Maxwell equations in a vacuum:"

$$\nabla \cdot \vec{E} = 0 \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

for some constant $c^2 > 0$. Prove that these vector fields satisfy the "wave equations"

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} \qquad \nabla^2 \vec{B} = \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2}.$$

Here $\nabla^2 \vec{E}$ is the vector Laplacian

$$\nabla^2 \vec{E} = \frac{\partial^2 \vec{E}}{\partial x^2} + \frac{\partial^2 \vec{E}}{\partial y^2} + \frac{\partial^2 \vec{E}}{\partial z^2},$$

and $\nabla^2 \vec{B}$ is defined similarly (with $\vec{E}$ replaced by $\vec{B}$).

By completing this exercise, you are showing that the fundamental laws of electrodynamics suggest the possibility of electromagnetic waves, i.e. light. *Fiat lux!*