Fall 2025 **Math 53, Sections 104/106**

Discussion #24

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1. Compute the following line integrals:

   (a) $\int_C y^2\,\mathrm{d}x + x^2\,\mathrm{d}y$ where C is the line segment from $(1,0)$ to $(4,1)$.
   We can parametrize C by $x = 1+3t, y = t, 0 \le t \le 1$. Hence

   $$\int_C y^2\,\mathrm{d}x + x^2\,\mathrm{d}y = \int_0^1 t^2 \cdot 3\,\mathrm{d}t + (1+3t)^2\,\mathrm{d}t = \int_0^1 12t^2 + 6t + 1\,\mathrm{d}t = 4+3+1 = 8.$$

   (b) $\int_C x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z$ where C is the straight line connecting $(0,0,0)$ to $(1,2,3)$. Can you figure out what the integral will be when the endpoint of C is an arbitrary point (x_0, y_0, z_0) ?
   We can parametrize this path by $x = t, y = 2t, z = 3t, 0 \le t \le 1$ obtaining

   $$\int_C x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z = \int_0^1 t\,\mathrm{d}t + 2t\cdot 2\,\mathrm{d}t + 3t \cdot 3\,\mathrm{d}t = \int_0^1 14t\,\mathrm{d}t = 7.$$

   For the general case we can use the parametrization $x = x_0 t, y = y_0 t, z = z_0 t, 0 \le t \le 1$ and redoing the calculation above shows that

   $$\int_C x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z = \int_0^1 ({x_0}^2 + {y_0}^2 + {z_0}^2)t\,\mathrm{d}t = \frac{1}{2}({x_0}^2 + {y_0}^2 + {z_0}^2)$$

   (c) $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$ where $\vec{F}(x,y) = (y,1)$ and C is the unit circle, traversed counterclockwise. Can you say something about the integral of $\vec{F}_2(x,y) = (y,0)$ along the same curve without doing another computation?
   We parametrize the circle by $\vec{r}(t) = \langle \cos t, \sin t\rangle, 0 \le t \le 2\pi$ and obtain using $\vec{r}'(t) = \langle -\sin t, \cos t\rangle$:

   $$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_0^{2\pi} -\sin^2 t + \cos t\,\mathrm{d}t = -\frac{1}{2}2\pi = -\pi$$

   We don't need to compute $\int_C \vec{F}_2 \cdot \mathrm{d}\vec{r}$ again explicitly because $\vec{F}_2(x,y) - \vec{F}(x,y) = \langle 0,1\rangle = \nabla f$ for $f(x,y) = y$ is a conservative vector field. Hence $\int_C \vec{F}_2 \cdot \mathrm{d}\vec{r} = \int_C \vec{F} \cdot \mathrm{d}\vec{r} + \int_C \nabla f \cdot \mathrm{d}\vec{r} = \pi + f(1,0) - f(1,0) = \pi$.

2. Use Green's theorem to convert each of the following line integrals $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$ to double integrals. Then evaluate. All curves C are oriented counterclockwise.

(a) C is the ellipse $x^2 + y^2/4 = 1$ and $\vec{F}(x,y) = \langle 2x - y, 3x + 2y \rangle$.
We have $\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \iint_D Q_x - P_y = \iint_D 4\,\mathrm{d}x\,\mathrm{d}y = 4 \cdot \mathrm{area(D)} = 8\pi$. Here and in all further solutions in this section, D is the region enclosed by C.

(b) C is the circle $x^2 + y^2 = 1$ and $\vec{F}(x,y) = \frac{1}{3}\langle -y^3, x^3 \rangle$.
By Green's theorem, the line integral equals $\iint_{x^2+y^2 \le 1} x^2 + y^2\,\mathrm{d}x\,\mathrm{d}y$. This integral is simplest in polar coordinates, where it becomes $\int_0^1 \int_0^{2\pi} r^3 d\theta dr = \pi/2$.

(c) C is the triangle with vertices at $(0,0), (1,0), (0,1)$ and $\vec{F}(x,y) = \langle x^2 y, e^{y^2} + x \rangle$.
By Green's theorem, the line integral equals $\iint_D 1 - x^2\,\mathrm{d}x\,\mathrm{d}y$. Firstly, $\iint_D \mathrm{d}x\,\mathrm{d}y = 1/2$. Next,

$$\iint_D x^2\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \int_0^{1-y} x^2\,\mathrm{d}x\,\mathrm{d}y = 1/12,$$

so altogether, the integral is $1/2 - 1/12 = 5/12$.

3. Let C be a simple, positively oriented, closed curve in $\mathbb{R}^2$. Using Green's theorem, check that $\int_C f(x)\,\mathrm{d}x + g(y)\,\mathrm{d}y = 0$ for arbitrary smooth functions f, g. Can you give an explanation without Green's theorem?

   The line integral is zero by Green's theorem since $Q_x - P_y = 0$ in this situation. Without Green's theorem, let $\vec{r}(t) = \langle x(t), y(t) \rangle$ be a parameterization of C, with $0 \le t \le T$. Then, $\int_C f(x)\,\mathrm{d}x = \int_0^T f(x(t))x'(t)dt$. With the substitution $u = x(t)$, we see that this integral is zero, since $x(0) = x(T)$. The same is true for the other half of the integral. (Alternatively, note that $\langle f(x), g(y) \rangle$ is conservative, since you may integrate each component separately to find a potential.)

4. Consider the non-standard parameterization of the unit circle $x = \sin(t), y = \cos(t)$ with $0 \le t \le 2\pi$. Check that $\int_C x\,\mathrm{d}y$ is not the area enclosed by C, as "promised" by Green's Theorem. What went wrong?

   We have $\int_C x\,\mathrm{d}y = \int_0^{2\pi} -\sin^2(t)dt = -\pi$ but the area enclosed by C is π. The problem is that this parameterization of the unit circle is oriented clockwise and Green's theorem requires a counterclockwise orientation. Reversing the direction of a path negates a line integral over that path.

5. Consider the following about a special vector field.

   (a) Let C be the square centered at the origin with side length 4, oriented counterclockwise. Compute $\int_C \frac{-y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y$. It will help to know that the vector field $\langle P, Q \rangle = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle$ satisfies $P_y = Q_x$ everywhere except the origin but is not conservative because $\int_\gamma P\,\mathrm{d}x + Q\,\mathrm{d}y = 2\pi$ where γ is the unit circle, oriented counterclockwise.
   If D is the region between C and γ, then $\int_D Q_x - P_y\,\mathrm{d}x\,\mathrm{d}y = 0$ since the integrand is zero, so $0 = \int_C \frac{-y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y - \int_\gamma \frac{-y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y$, so we find that the desired integral is 2π.

   (b) Let $\vec{F}$ be the vector field in the previous part. Explain why, using Green's theorem, if C is a simple positively oriented curve contained in the upper half plane $y > 0$, then $\int_C \vec{F} \cdot d\vec{r} = 0$.

Since C is a simple curve in the upper half plane, it bounds a region in the upper half plane, where the vector field is defined (since $(x, y) \neq (0, 0)$). Using the fact that $Q_x = P_y$ for this vector field, Green's theorem tells us that $\int_C \vec{F} \cdot d\vec{r} = 0$. In fact, the hypothesis that C is simple is not needed.

All problems courtesy of Carlos Esparza.