Fall 2025 **Math 53, Sections 104/106**

Discussion #24

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1. Compute the following line integrals:

   (a) $\int_C y^2\,\mathrm{d}x + x^2\,\mathrm{d}y$ where C is the line segment from $(1,0)$ to $(4,1)$.

   (b) $\int_C x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z$ where C is the straight line connecting $(0,0,0)$ to $(1,2,3)$. Can you figure out what the integral will be when the endpoint of C is an arbitrary point (x_0, y_0, z_0) ?

   (c) $\int_C \vec{F}\cdot \mathrm{d}\vec{r}$ where $\vec{F}(x,y) = (y,1)$ and C is the unit circle, traversed counterclockwise. Can you say something about the integral of $\vec{F}_2(x,y) = (y,0)$ along the same curve without doing another computation?

2. Use Green's theorem to convert each of the following line integrals $\int_C \vec{F}\cdot \mathrm{d}\vec{r}$ to double integrals. Then evaluate. All curves C are oriented counterclockwise.

   (a) C is the ellipse $x^2 + y^2/4 = 1$ and $\vec{F}(x,y) = \langle 2x - y, 3x + 2y\rangle$.

   (b) C is the circle $x^2 + y^2 = 1$ and $\vec{F}(x,y) = \frac{1}{3}\langle -y^3, x^3\rangle$.

   (c) C is the triangle with vertices at $(0,0), (1,0), (0,1)$ and $\vec{F}(x,y) = \langle x^2 y, e^{y^2} + x\rangle$.

3. Let C be a simple, positively oriented, closed curve in $\mathbb{R}^2$. Using Green's theorem, check that $\int_C f(x)\,\mathrm{d}x + g(y)\,\mathrm{d}y = 0$ for arbitrary smooth functions f, g. Can you give an explanation without Green's theorem?

4. Consider the non-standard parameterization of the unit circle $x = \sin(t), y = \cos(t)$ with $0 \leq t \leq 2\pi$. Check that $\int_C x\,\mathrm{d}y$ is not the area enclosed by C, as "promised" by Green's Theorem. What went wrong?

5. Consider the following about a special vector field.

   (a) Let C be the square centered at the origin with side length 4, oriented counterclockwise. Compute $\int_C \frac{-y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y$. It will help to know that the vector field $\langle P, Q\rangle = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\rangle$ satisfies $P_y = Q_x$ everywhere except the origin but is not conservative because $\int_\gamma P\,\mathrm{d}x + Q\,\mathrm{d}y = 2\pi$ where γ is the unit circle, oriented counterclockwise.

   (b) Let $\vec{F}$ be the vector field in the previous part. Explain why, using Green's theorem, if C is a simple positively oriented curve contained in the upper half plane $y > 0$, then $\int_C \vec{F} \cdot d\vec{r} = 0$.