Fall 2025 **Math 53, Sections 104/106**

Discussion #23

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Date: October 24

1. For each of the following vector fields $\vec{F}$, either prove that $\vec{F}$ is conservative by finding a function f such that $\nabla f = \vec{F}$, or prove that f is not conservative.

   (a) $\vec{F}(x,y) = x\,\vec{\imath} + y\,\vec{\jmath}$
   This is conservative. To see this, write $\vec{F} = \nabla f$, and try to guess f. We get $f_x = x$ and $f_y = y$. Hence f_x has antiderivative $\frac{1}{2}x^2$ and f_y has antiderivative $\frac{1}{2}y^2$. These are not the same but we can add them and since each variable only appears in one of the atiderivatives, $f(x,y) = \frac{1}{2}(x^2+y^2)$ satisfies for conditions.

   (b) $\vec{F}(x,y) = x\,\vec{\imath} + x\,\vec{\jmath}$
   This is not conservative. If it were, we could write $\vec{F} = \nabla f$ for some $f(x,y)$. We would then have $f_x = x$ and $f_y = x$. But then we could compute
   $$f_{xy} = 0 \neq 1 = f_{yx},$$
   contradicting Clairaut's theorem.

   (c) $\vec{F}(x,y,z) = yz\,\vec{\imath} + xz\,\vec{\jmath} + xy\,\vec{k}$
   This is conservative. To see that, we need to find an f such that $f_x = yz, f_y = xz$ and $f_z = xy$. But integrating each of these produces $xyz + C$ as a possible antiderivative, hence $f(x,y,z) = xyz$ is a potential for $\vec{F}$.

   (d) $\vec{F} = xz\,\vec{\imath} + yz\,\vec{\jmath} + xy\,\vec{k}$
   This is not conservative. If we write $\vec{F} = \nabla f$, then we compute $f_{xz} = x$, but $f_{zx} = y$, contradicting Clairaut's theorem.

2. Compute the following line integrals:

   (a) $\int_C x\,\mathrm{d}s$ where C is the graph of $f(x) = \frac{1}{2}x^2$ going from $x = 0$ to $x = 2$.
   We can parametrize the graph using $x = t, y = f(t) = \frac{1}{2}t^2, 0 \leq t \leq 2$. Using this we get $\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{1+t^2}$. Now we compute, using the substitution $u = 1+t^2$, $\mathrm{d}u = 2t\,\mathrm{d}t$
   $$\int_C x\,\mathrm{d}s = \int_0^2 t\sqrt{1+t^2}\,\mathrm{d}t = \frac{1}{2}\int_1^5 \sqrt{u}\,\mathrm{d}u = \frac{1}{2}\frac{2}{3}(5^{3/2} - 1^{3/2}) = \frac{5\sqrt{5}-1}{3}$$

   (b) $\int_C xy^4\,\mathrm{d}s$ where C is the right half of the unit circle.
   We know that C can be parametrized by $x = \cos t, y = \sin t, -\pi/2 \leq t \leq \pi/2$. Now using the definition of line integrals and $\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{\sin^2 t + \cos^2 t}$ we obtain
   $$\int_C xy^4\,\mathrm{d}s = \int_{-\pi/2}^{\pi/2} \cos t \sin^4 t\,\mathrm{d}t = \int_{-1}^{1} u^4\,\mathrm{d}u = \frac{2}{5}$$

(c) $\int_C x^2 y\, \mathrm{d}s$ in 3D where C is given by $x = \cos t, y = \sin t, z = t, 0 \le t \le \pi/2$.
First compute $\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} = \sqrt{\sin^2 t + \cos^2 t + 1^2} = \sqrt{2}$. Now we ca just compute

$$\int_C x^2 y\, \mathrm{d}s = \int_0^{\pi/2} \cos^2 t \sin t \sqrt{2}\, \mathrm{d}t = \sqrt{2} \int_0^1 u^2\, \mathrm{d}u = \frac{\sqrt{2}}{3}$$

3. Compute the following line integrals:

(a) $\int_C y^2\, \mathrm{d}x + x^2\, \mathrm{d}y$ where C is the line segment from $(1, 0)$ to $(4, 1)$.
We can parametrize C by $x = 1 + 3t, y = t, 0 \le t \le 1$. Hence

$$\int_C y^2\, \mathrm{d}x + x^2\, \mathrm{d}y = \int_0^1 t^2 \cdot 3\, \mathrm{d}t + (1+3t)^2\, \mathrm{d}t = \int_0^1 12t^2 + 6t + 1\, \mathrm{d}t = 4 + 3 + 1 = 8$$

(b) $\int_C x\, \mathrm{d}x + y\, \mathrm{d}y + z\, \mathrm{d}z$ where C is the straight line connecting $(0, 0, 0)$ to $(1, 2, 3)$. Can you figure out what the integral will be when the endpoint of C is an arbitrary point (x_0, y_0, z_0) ?
We can parametrize this path by $x = t, y = 2t, z = 3t, 0 \le t \le 1$ obtaining

$$\int_C x\, \mathrm{d}x + y\, \mathrm{d}y + z\, \mathrm{d}z = \int_0^1 t\, \mathrm{d}t + 2t \cdot 2\, \mathrm{d}t + 3t \cdot 3\, \mathrm{d}t = \int_0^1 14t\, \mathrm{d}t = 7$$

For the general case we can use the parametrization $x = x_0 t, y = y_0 t, z = z_0 t, 0 \le t \le 1$ and redoing the calculation above shows that

$$\int_C x\, \mathrm{d}x + y\, \mathrm{d}y + z\, \mathrm{d}z = \int_0^1 ({x_0}^2 + {y_0}^2 + {z_0}^2) t\, \mathrm{d}t = \frac{1}{2}({x_0}^2 + {y_0}^2 + {z_0}^2)$$

(c) $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$ where $\vec{F}(x, y) = (y, 1)$ and C is the unit circle, traversed counterclockwise. Can you say something about the integral of $\vec{F}_2(x, y) = (y, 0)$ along the same curve without doing another computation?
We parametrize the circle by $\vec{r}(t) = \langle \cos t, \sin t \rangle, 0 \le t \le 2\pi$ and obtain using $\vec{r}'(t) = \langle -\sin t, \cos t \rangle$:

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_0^{2\pi} -\sin^2 t + \cos t\, \mathrm{d}t = -\frac{1}{2} 2\pi = -\pi$$

We don't need to compute $\int_C \vec{F}_2 \cdot \mathrm{d}\vec{r}$ again explicitly because $\vec{F}_2(x, y) - \vec{F}(x, y) = \langle 0, 1 \rangle = \nabla f$ for $f(x, y) = y$ is a conservative vector field. Hence $\int_C \vec{F}_2 \cdot \mathrm{d}\vec{r} = \int_C \vec{F} \cdot \mathrm{d}\vec{r} + \int_C \nabla f \cdot \mathrm{d}\vec{r} = \pi + f(1, 0) - f(1, 0) = \pi$.

All problems courtesy of Carlos Esparza.