Fall 2025 **Math 53, Sections 104/106**

Discussion #22

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1. Describe the following surfaces (defined by Cartesian coordinates) in terms of spherical coordinates.

   $x = \sqrt{3}y$.

   This is equivalent to $\theta = \arctan(1/\sqrt{3}) = \pi/6$ OR $\theta = 7\pi/6$.

   $z^2 = x^2 + y^2$.

   By geometric reasoning, the answer is $\phi = \pi/4$ or $\phi = 3\pi/4$. This may also be done algebraically.

   $x^2 + y^2 + z^2/4 = 1$.

   By substitution, we get $\rho^2 \sin^2\phi + \rho^2\cos^2\phi/4 = 1$.

2. Find the volume of the region bounded by the sphere $x^2 + y^2 + z^2 = 4$ and the plane $z = 1$.

   The region enclosed is given by $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi/3$ and $\sec\phi \le \rho \le 2$. So, the volume is (using $\sec'(\phi) = \sec\phi\tan\phi$):

$$\int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^2 \rho^2\sin\phi d\rho d\phi d\theta = 2\pi\int_0^{\pi/3}\frac{1}{3}\sin(\phi)(8-\sec^3(\phi))d\phi = 5\pi/3.$$

3. Let R be the region lying above the cone $z^2 = x^2 + y^2$ and below the unit sphere. Compute

$$\iiint_R z^2\,\mathrm{d}V.$$

   The region is described by $0 \le \rho \le 1, 0 \le \phi \le \pi/4$ and $0 \le \pi \le 2\pi$, so

$$\begin{aligned}\iiint_R z^2\,\mathrm{d}V &= \int_0^1\int_0^{\pi/4}\int_0^{2\pi}(\rho\cos\phi)^2\sin\phi\rho^2\,\mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}\rho \\ &= 2\pi\int_0^1\rho^4\int_0^{\pi/4}\sin\phi\cos^2\phi\,\mathrm{d}\phi\,\mathrm{d}\rho \\ &= \frac{2\pi}{5}\frac{1}{12}(4-\sqrt{2}) = \frac{\pi}{60}(4-\sqrt{2})\end{aligned}$$

4. Find the absolute value of the Jacobian determinant for each of the following changes of coordinates.

(a) $x = au + bv$ and $y = cu + dv$.

We take the determinant of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, which is $ad - bc$, so take absolute values to get $|ad - bc|$.

(b) $x = u^2 - v^2$ and $y = 2uv$.

We compute $\det\begin{pmatrix} 2u & -2v \\ 2v & 2u \end{pmatrix} = 4(u^2 + v^2)$, which is non-negative, so we do not need absolute values.

(c) $x = e^u \cos(v)$ and $y = e^u \sin(v)$.

We compute $\det\begin{pmatrix} e^u \cos(v) & -e^u \sin(v) \\ e^u \sin(v) & e^u \cos(v) \end{pmatrix} = e^{2u}$. This is always positive.

(d) $x = \frac{u}{u^2+v^2}$ and $y = \frac{-v}{u^2+v^2}$. Note that this transformation is its own inverse, in the sense that we can solve $u = \frac{x}{x^2+y^2}$ and $v = \frac{-y}{x^2+y^2}$. Also check that $(x^2 + y^2)(u^2 + v^2) = 1$.

We compute $\det\begin{pmatrix} \frac{1}{u^2+v^2} - \frac{2u^2}{(u^2+v^2)^2} & \frac{-2uv}{(u^2+v^2)^2} \\ \frac{2uv}{(u^2+v^2)^2} & \frac{-1}{u^2+v^2} + \frac{2v^2}{(u^2+v^2)^2} \end{pmatrix} = \frac{1}{(u^2+v^2)^2}$, which, as before, is always positive.

5. Here are some change of variable integration problems.

(a) Consider the following region $\mathcal{R}$ in the plane: $3x^2 + 4xy + 3y^2 \leq 1$. Describe the transformed region using the change of variables $x = v - u$ and $y = u + v$, and find its area.

We have $3(v-u)^2 + 4(v^2 - u^2) + 3(u+v)^2 = 2u^2 + 10v^2 \leq 1$. This is an ellipse with axes of length $1/\sqrt{2}$ and $1/\sqrt{10}$, so has area $\pi/\sqrt{20}$. The Jacobian of this transformation is 2. Now, $A = \int_{\mathcal{R}} dx\, dy = \int_{\mathcal{R}'} 2\, du\, dv = 2\pi/\sqrt{20} = \pi/\sqrt{5}$, where $\mathcal{R}'$ is the ellipse in the uv-plane.

(b) Let D be the annulus $1 \leq x^2 + y^2 \leq 4$ and consider the integral

$$\iint_D \frac{1}{(x^2+y^2)^2} e^{\frac{x}{x^2+y^2}}\, dx\, dy.$$

Perform the change of variables $x = \frac{u}{u^2+v^2}, y = \frac{-v}{u^2+v^2}$ to simplify the integral, but do not evaluate.

From our work on this substitution in the first problem, we know that the new region of integration is $D' : 1/4 \leq u^2 + v^2 \leq 1$ and the integral becomes

$$\iint_{D'} (u^2+v^2)^2 e^u \frac{1}{(u^2+v^2)^2}\, du\, dv = \iint_{D'} e^u\, du\, dv.$$

All problems courtesy of Carlos Esparza.