Fall 2025 **Math 53, Sections 104/106**

Discussion #22

GSI: Zack Stier
Date: October 22

1. Describe the following surfaces (defined by Cartesian coordinates) in terms of spherical coordinates.

   $x = \sqrt{3}y$.
   $z^2 = x^2 + y^2$.
   $x^2 + y^2 + z^2/4 = 1$.

2. Find the volume of the region bounded by the sphere $x^2 + y^2 + z^2 = 4$ and the plane $z = 1$.

3. Let R be the region lying above the cone $z^2 = x^2 + y^2$ and below the unit sphere. Compute
$$\iiint_R z^2 \, \mathrm{d}V.$$

4. Find the absolute value of the Jacobian determinant for each of the following changes of coordinates.

   (a) $x = au + bv$ and $y = cu + \mathrm{d}v$.
   (b) $x = u^2 - v^2$ and $y = 2uv$.
   (c) $x = e^u \cos(v)$ and $y = e^u \sin(v)$.
   (d) $x = \frac{u}{u^2+v^2}$ and $y = \frac{-v}{u^2+v^2}$. Note that this transformation is its own inverse, in the sense that we can solve $u = \frac{x}{x^2+y^2}$ and $v = \frac{-y}{x^2+y^2}$. Also check that $(x^2 + y^2)(u^2 + v^2) = 1$.

5. Here are some change of variable integration problems.

   (a) Consider the following region $\mathcal{R}$ in the plane: $3x^2 + 4xy + 3y^2 \leq 1$. Describe the transformed region using the change of variables $x = v - u$ and $y = u + v$, and find its area.
   (b) Let D be the annulus $1 \leq x^2 + y^2 \leq 4$ and consider the integral
   $$\iint_D \frac{1}{(x^2 + y^2)^2} e^{\frac{x}{x^2+y^2}} \, \mathrm{d}x \, \mathrm{d}y.$$
   Perform the change of variables $x = \frac{u}{u^2+v^2}, y = \frac{-v}{u^2+v^2}$ to simplify the integral, but do not evaluate.

All problems courtesy of Carlos Esparza.