**Fall 2025** **Math 53, Sections 104/106**

# Discussion #19/21

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1. $\iint_D x^2 y \, dA$ where $D$ is the top half of the disk with center the origin and radius 5;
   The region is $D = \{(r,\theta) \mid 0 \le r \le 5, 0 \le \theta \le \pi\}$. Then
   $$\iint_D x^2 y \, dA = \int_0^\pi \int_0^5 (r\cos\theta)^2 r \sin\theta r \, dr \, d\theta = \left(\int_0^\pi \cos^2\theta \sin\theta \, d\theta\right)\left(\int_0^5 r^4 \, dr\right) = \frac{1250}{3}.$$

2. $\iint_D e^{-x^2-y^2} \, dA$ where $D$ is the region bounded by the semicircle $x = \sqrt{4-y^2}$ and the $y$-axis.
   $$\iint_D e^{-x^2-y^2} \, dA = \int_{-\pi/2}^{\pi/2} \int_0^2 e^{-r^2} r \, dr \, d\theta = \pi\left(-\frac{1}{2}e^{-r^2}|_0^2\right) = \frac{\pi}{2}(1-e^{-4}).$$

3. Evaluate $\iint_D (y^2 + 3x) \, dA$, where $D$ is the region in the fourth quadrant between $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.
   The two circles are $r = 1$ and $r = 2$, and the fourth quadrant is $\theta \in [\frac{3\pi}{2}, 2\pi]$. Substituting $x = r\cos\theta$, $y = r\sin\theta$, and $dA = r \, dr \, d\theta$, we get
   $$\int_{\frac{3\pi}{2}}^{2\pi} \int_1^2 (r^2 \sin^2\theta + 3r\cos\theta) r \, dr \, d\theta = \int_{\frac{3\pi}{2}}^{2\pi} \int_1^2 (r^3 \sin^2\theta + 3r^2\cos\theta) \, dr \, d\theta$$
   $$= \int_{\frac{3\pi}{2}}^{2\pi} \left[\frac{1}{4}r^4 \sin^2\theta + r^3\cos\theta\right]_1^2 d\theta = \int_{\frac{3\pi}{2}}^{2\pi} \left(\frac{15}{4}\sin^2\theta + 7\cos\theta\right) d\theta = \left[\frac{15}{8}\theta - \frac{15}{16}sin(2\theta) + 7\sin\theta\right]_{\frac{3\pi}{2}}^{2\pi}$$
   $$= \left(\frac{15}{4}\pi\right) - \left(\frac{45}{16}\pi - 7\right) = \frac{15}{16}\pi + 7$$

4. Evaluate the following integral:
   $$\int_0^3 \int_{-\sqrt{9-x^2}}^0 e^{x^2+y^2} \, dy \, dx$$
   We have $x^2 + y^2 = r^2$, so the integrand becomes $re^{r^2} \, dr \, d\theta$ (where the extra $r$ comes from the change of coordinates). The region described by this function is the part of the circle with radius 3 centered at the origin in the fourth quadrant; this is described by $r \le 3$ and $\frac{3\pi}{2} \le \theta \le 2\pi$. Thus the integral is
   $$\int_{\frac{3\pi}{2}}^{2\pi} \int_0^3 re^{r^2} \, dr \, d\theta = \int_{\frac{3\pi}{2}}^{2\pi} = \int_{\frac{3\pi}{2}}^{2\pi} \left[\frac{1}{2}e^{r^2}\right]_0^3 d\theta = \frac{1}{2}\int_{\frac{3\pi}{2}}^{2\pi} (e^9 - 1) \, d\theta = \frac{1}{2}\left[(e^9-1)\theta\right]_{\frac{3\pi}{2}}^{2\pi} = \frac{1}{4}(e^9-1)\pi.$$

5. Parameterize the following surfaces in an appropriate way (if they are not already parametrized) and compute their normal vectors and area.

(a) The portion of the elliptic paraboloid $z = x^2 + y^2$ lying over the unit disk.
This surface is the graph of $f(x, y) = x^2 + y^2$, so we know that

$$\vec{N} = \langle 1, 0, f_x \rangle \times \langle 0, 1, f_y \rangle = \langle -f_x, -f_y, 1 \rangle = \langle -2x, -2y, 1 \rangle.$$

The area is computed by the following integral over the unit disk $D$, which we compute in polar coordinates and using the substitution $u = 1 + 4r^2$

$$\begin{aligned}\int_D |\vec{N}| \, \mathrm{d}A &= \int_D \sqrt{1 + 4x^2 + 4y^2} \, \mathrm{d}A = \int_0^{2\pi} \int_0^1 \sqrt{1 + 4r^2} \, r \, \mathrm{d}r \, \mathrm{d}\phi \\ &= 2\pi \int_1^5 \sqrt{u} \frac{1}{8} \, \mathrm{d}u = \frac{\pi}{6} \left( 5^{3/2} - 1 \right)\end{aligned}$$

(b) The part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$.
This is the graph of $f(x, y) = xy$, so

$$\vec{N} = \langle -f_x, -f_y, 1 \rangle = \langle y, x, 1 \rangle.$$

Restricting the surface to the part inside the cylinder corresponds to restricting the domain of $f$ to the unit disk $D$. The area of the surface is given by

$$A = \iint_D \sqrt{1 + x^2 + y^2} \, \mathrm{d}A = \int_0^{2\pi} \int_0^1 \sqrt{r^2 + 1} r \, \mathrm{d}r \, \mathrm{d}\theta = \frac{2\pi}{3} \left( 2\sqrt{2} - 1 \right)$$

(The computation of the integral is analogous to problem 5a).

(c) The portion of $z = 2x^2 + 2y^2 - 7$ that lies inside the cylinder $x^2 + y^2 = 4$. We will do this as a polar integral. The integrand is $\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2 + 1} = \sqrt{(4x)^2 + (4y)^2 + 1} = \sqrt{16(x^2 + y^2) + 1} = \sqrt{16r^2 + 1}$. Thus the integral is

$$\int_0^{2\pi} \int_0^2 r\sqrt{16r^2 + 1} \, \mathrm{d}r \, \mathrm{d}\theta = \int_0^{2\pi} \mathrm{d}\theta \int_0^2 r\sqrt{16r^2 + 1} \, \mathrm{d}r = 2\pi \left[ \frac{1}{48} (16r^2 + 1)^{3/2} \right]_0^2 = \frac{\pi}{24} (65^{3/2} - 1).$$

(d) The surface area of the portion of $z = 2 - \sqrt{x^2 + y^2}$ above $z = 0$.
We will again do this as a polar integral. The surface meets $z = 0$ when $r = \sqrt{x^2 + y^2} = 2$. We have $\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{x^2+y^2}}$ and $\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{x^2+y^2}}$. Therefore the integrand is $\sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1} = \sqrt{2}$. We thus have

$$\int_0^{2\pi} \int_0^2 \sqrt{2} \, \mathrm{d}r \, \mathrm{d}\theta = \int_0^{2\pi} \mathrm{d}\theta \int_0^2 \sqrt{2} \, \mathrm{d}r = 4\sqrt{2}\pi.$$

6. Rewrite the integral
$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\,\mathrm{d}z\mathrm{d}y\,\mathrm{d}x$$
as the equivalent iterated integral in the five other orders.

$$\begin{aligned}\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x &= \int_0^1 \int_0^{y^2} \int_0^{1-y} f(x,y,z)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^1 \int_0^{1-z} \int_0^{y^2} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \int_0^1 \int_0^{1-y} \int_0^{y^2} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y \\ &= \int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(x,y,z)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x \\ &= \int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f(x,y,z)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z\end{aligned}$$

Problems 1, 2, 5(a), 5(b), and 6 courtesy of Carlos Esparza. Problems 3, 4, 5(c), and 5(d) courtesy of Peter Rowley.