Fall 2025 **Math 53, Sections 104/106**

Discussion #1

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1. Find Cartesian equations $f(x, y) = 0$ for the following parametrized curves.

   (a) $x = \sqrt{t+1}, y = \frac{1}{t+1}$, for $t > -1$.

   (b) $x = 4 - 2t, y = 3 + 6t - 4t^2$.

   (c) $x = 2e^t, y = \cos(1 + e^{3t})$.

   (a) We have

   $$x^2 = t + 1 = \frac{1}{y},$$

   so the Cartesian equation is $x^2 = 1/y$. Note that the constraint $t > -1$ means that $x > 0$ and $y > 0$.

   (b) We can solve the x equation to see $t = 2 - \frac{1}{2}x$. We then plug this in to the y equation to get

   $$\begin{aligned} y &= 3 + 6\left(2 - \frac{1}{2}x\right) - 4\left(2 - \frac{1}{2}x\right)^2 \\ &= 3 + 12 - 3x - 16 + 8x - x^2 \\ &= -1 + 5x - x^2, \end{aligned}$$

   which is a Cartesian equation for the curve.

   (c) Note that $e^t = x/2$, so

   $$y = \cos(1 + (e^t)^3) = \cos(1 + x^3/8).$$

   The outer sides of this give a Cartesian equation for the curve (which holds for $x > 0$, since e^t takes on all positive values).

2. Compute the slopes of the following curves at a point in time t. Find the points where the tangents are vertical and horizontal and compute the second derivative d^2y/dx^2 at the horizontal points.

   (a) $x = \cos t, y = \sin t$.

   (b) $x = t^2 - 1$ $y = t^3 - t$.

   (c) $x = e^t - 1, y = \sin t$.

(a) $x'(t) = -\sin t, y'(t) = \cos t$ so the slope is $\frac{dy}{dx} = -\frac{\cos t}{\sin t} = -\cot t$. The tangents are horizontal where $x'(t) = 0$, i.e. at $t = \pi/2 + k\pi$ and vertical where $y'(t) = 0$, i.e. at $k\pi, k \in \mathbb{Z}$. For the second derivative, we get

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{1}{x'(t)} \frac{\mathrm{d}}{\mathrm{d}t} \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-1}{\sin t} \left(\frac{1}{\tan^2 t \cos^2 t} \right) = \frac{-1}{\sin^3 t}$$

which is -1 at $t = \pi/2 + k\pi$ for even k and $+1$ for odd k.

(b) $x'(t) = 2t, y'(t) = 3t^2 - 1$ so the slope is $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3t^2-1}{2t}$. The tangents are horizontal at $t = \pm 1/\sqrt{3}$ and vertical at $t = 0$. Note that the point $(0, 0)$ which corresponds to $t = \pm 1$ has two different tangents. The second derivative is

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{1}{2t} \frac{\mathrm{d}}{\mathrm{d}t} \frac{3t^2 - 1}{2t} = \frac{12t^2 - 2(3t^2 - 1)}{8t^3} = \frac{6t^2 + 2}{8t^3}$$

So at the horizontal points we get $\pm 3\sqrt{3}/2$.

(c) $x'(t) = e^t, y'(t) = \cos t$ so the slope is $\frac{\mathrm{d}y}{\mathrm{d}x} = e^{-t} \cos t$. The tangents are horizontal at $t = \pi/2 + k\pi, k \in \mathbb{Z}$ and never vertical. The second derivative is

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = e^{-t} \frac{\mathrm{d}}{\mathrm{d}t} e^{-t} \cos t = -e^{-2t} (\sin t + \cos t)$$

At the points with horizontal tangents we get $(-1)^{k+1} e^{-\pi - 2\pi k}$.

3. Find the length of each curve.

(a) $x = 1 + 3t^2, y = 4 + 2t^3, 0 \leqslant t \leqslant 1$.

(b) $x = e^t - t, y = 4e^{t/2}, 0 \leqslant t \leqslant 2$.

(c) $x = e^t \cos t, y = e^t \sin t, 0 \leqslant t \leqslant \pi$.

(a) Applying the formula we have

$$\begin{aligned} L = \int_0^1 \sqrt{(6t)^2 + (6t^2)^2}\, \mathrm{d}t &= \int_0^1 6t\sqrt{1 + t^2}\, \mathrm{d}t \\ &= [2(1 + t^2)^{3/2}]_0^1 \\ &= 4\sqrt{2} - 2. \end{aligned}$$

(b) Applying the formula we have

$$L = \int_0^2 \sqrt{(e^t - 1)^2 + (2e^{t/2})^2}\, \mathrm{d}t = \int_0^2 \sqrt{e^{2t} + 2e^t + 1}\, \mathrm{d}t = \int_0^2 e^t + 1\, \mathrm{d}t = e^2 + 1.$$

(c) Applying the formula we have

$$\begin{aligned} L = \int_0^\pi \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \cos t + e^t \sin t)^2}\, \mathrm{d}t &= \int_0^\pi \sqrt{2e^{2t} \cos^2 t + 2e^{2t} \sin^2 t}\, \mathrm{d}t \\ &= \int_0^\pi \sqrt{2} e^t\, \mathrm{d}t \\ &= \sqrt{2}(e^\pi - 1). \end{aligned}$$

4. Convert the following from polar to Cartesian, or vice versa. What curve does it trace out?

   (a) $x + y = 1, 0 \leqslant y \leqslant 1$.

   (b) $r = -3, 0 \leqslant \theta \leqslant \pi$.

   (a) $r\sin\theta = y = 1 - x = 1 - r\cos\theta$, so $r = \frac{1}{\sin\theta + \cos\theta}$ for $0 \leqslant \theta \leqslant \frac{\pi}{2}$. This is a line segment connecting $(1, 0)$ to $(0, 1)$; when we trace it in increasing θ it is oriented to the top-left.

   (b) $x^2 + y^2 = 9$, in the lower half-plane (the semicircle $y = -\sqrt{9 - x^2}$).

5. Compute the slopes of the following curves. Find the points where the tangents are vertical and horizontal.

   (a) $r = 3\cos\theta$;

   (b) $r = 1 - \sin\theta$;

   (c) $r = \sec\theta$;

   (d) $r = e^\theta$.

   (a) $\mathrm{d}y/\,\mathrm{d}\theta = -3\sin^2\theta + 3\cos^2\theta$ and $\mathrm{d}x/\,\mathrm{d}\theta = -6\sin\theta\cos\theta$ so the slope is $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3\cos(2\theta)}{-3\sin(2\theta)} = -\tan(2\theta)$. The horizontal tangents are where $3\cos(2\theta) = 0$ which occurs at $\pi/4 + k\pi/2, k \in \mathbb{Z}$ and the vertical tangents are where $-3\sin(2\theta) = 0$ which occurs at $k\pi/2, k \in \mathbb{Z}$.

   (b) $\mathrm{d}y/\,\mathrm{d}\theta = \cos\theta - \sin(2\theta)$ and $\mathrm{d}x/\,\mathrm{d}\theta = -\sin\theta - \cos(2\theta)$ so the slope is $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos\theta - \sin(2\theta)}{-\sin\theta - \cos(2\theta)}$. The horizontal tangents are where $\cos\theta(1 - 2\sin\theta) = 0$ which occurs at $\theta = \pi/2 + k\pi, k \in \mathbb{Z}$ and $\theta = \pi/6 + 2k\pi, 5\pi/6 + 2l\pi, k, l \in \mathbb{Z}$. The vertical tangents are where $1 + \sin\theta - 2\sin^2\theta = (1 + 2\sin\theta)(1 - \sin\theta) = 0$ which occurs at $\pi/2 + 2k\pi, k \in \mathbb{Z}$ and $\theta = 7\pi/6 + 2k\pi, 11\pi/6 + 2l\pi, k, l \in \mathbb{Z}$. Notice there is overlap at $\theta = \pi/2 + 2k\pi$ so we must take a limit to verify the slope. We see that

$$\lim_{\theta \to \pi/2} \frac{\cos\theta - \sin(2\theta)}{-\sin\theta - \cos(2\theta)} = \lim_{\theta \to \pi/2} \frac{-\sin\theta - 2\cos(2\theta)}{-\cos\theta + 2\sin(2\theta)} = \frac{1}{0}$$

   so the horizontal tangents only occur at $\theta = \pi/2 + (2k+1)\pi, k \in \mathbb{Z}$ and $\theta = \pi/6 + 2k\pi, 5\pi/6 + 2l\pi, k, l \in \mathbb{Z}$.

   (c) $\mathrm{d}y/\,\mathrm{d}\theta = \sec\theta\tan\theta\sin\theta + \tan\theta$ and $\mathrm{d}x/\,\mathrm{d}\theta = \tan\theta - \sin\theta\sec\theta = 0$. The slope $\frac{\mathrm{d}y}{\mathrm{d}x}$ is undefined. So for all θ we have a vertical tangent and no horizontal tangents.

   (d) $\mathrm{d}y/\,\mathrm{d}\theta = e^\theta(\sin\theta + \cos\theta)$ and $\mathrm{d}x/\,\mathrm{d}\theta = e^\theta(\cos\theta - \sin\theta)$ so the slope is $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$. The horizontal tangents are where $\tan\theta = -1$ which occurs at $3\pi/4 + k\pi, k \in \mathbb{Z}$ and the vertical tangents are where $\tan\theta = 1$ which occurs at $\pi/4 + k\pi, k \in \mathbb{Z}$.

Problems 1–3 and 5 courtesy of Carlos Esparza.