Notes on Lee–Yang-type theorems

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Contents

1 Classical Lee–Yang

(Following notes of Srivastava)

1.1 Ferromagnetic Ising model

Definition 1.1 (ferromagnetic Ising model)**.** Consider a graph $G(V, E)$. For an assignment of spins $\sigma \in \{\pm\}^V$, say that

$$\begin{aligned} d(\sigma) &= \#\{uv \in E : \sigma(u) \neq \sigma(v)\} && \text{(number of edges that } \sigma \text{ cuts)}, \\ m(\sigma) &= \#\{v \in V : \sigma(v) = +\} && \text{(number of sites in the + half)}. \end{aligned}$$

Then for $0 < \beta < 1$, we give σ the weight $\beta^{d(\sigma)}\lambda^{m(\sigma)}$ and assign the partition function

$$Z_{\beta,G}(\lambda) = \sum_{\sigma \in \{\pm\}^V} \beta^{d(\sigma)}\lambda^{m(\sigma)}$$

to normalize the distribution on $\{\pm\}^V$.

β plays the role of *temperature*[*] and controls the extent to which we "punish" or "reward" the size of the cut. One extreme of this is $\beta = 0$ where the only surviving terms have no cut edges; if G is connected then this requires σ to be constant, and we call this a **frozen** configuration. The other extreme is $\beta = 1$ where neighbor interactions (cuts) have no impact, so sites act as IRVs. Meanwhile, λ is the **vertex activity**, which biases the sites to some degree in favor of $+$.

[*] Rather than inverse temperature

Consider the expected fraction $M_\beta(\lambda)$ of the graph in the $+$ half, which is

$$M_\beta(\lambda) = \frac{\sum_{\sigma\in\{\pm\}^V} \frac{m(\sigma)}{\#V}\beta^{d(\sigma)}\lambda^{m(\sigma)}}{\sum_{\sigma\in\{\pm\}^V} \beta^{d(\sigma)}\lambda^{m(\sigma)}} = \frac{1}{\#V Z_\beta(\lambda)}\left(z\frac{\partial}{\partial z}Z_\beta(z)\right)\bigg|_{z=\lambda} = -\frac{1}{\#V}\left(z\frac{\partial}{\partial z}\log Z_\beta(z)\right)\bigg|_{z=\lambda}. \tag{1.2}$$

Theorem 1.3 (Lee–Yang '52)
Let G be a finite graph, $0 < \beta < 1$, and $Z = Z_{\beta,G}$ the partition function of the ferromagnetic Ising model. Then, all of Z's zeroes lie on S^1.

Proof. Suppose each vertex v gets its own vertex activity λ_v which only appears when v is included in σ's $+$ half. Then the function to study is

$$Z(\lambda_v : v \in V) = Z(\boldsymbol{\lambda}) = \sum_{\sigma\in\{\pm\}^V} \beta^{d(\sigma)} \prod_{\substack{v\in V\\ \sigma(v)=+}} \lambda_v.$$

We shall show that if $|\lambda_v| > 1$ for all v, then $Z(\boldsymbol{\lambda}) \neq 0$.

We quickly remark why this proves the theorem. First, if $\boldsymbol{\lambda} = \lambda\mathbf{1}$ then we see that $Z(\lambda) \neq 0$ on $\lambda \notin \overline{\mathbb{D}}$. Then, note the algebraic fact that $Z(\lambda) = \lambda^{\#V}Z(1/\lambda)$ under the bijection of spins $\sigma \longmapsto \neg\sigma$ (flipping each spin does not change d).

Moving on to the proof of the new fact, we use the **Asano contraction**, where we quotient the pair of vertices $\{u, v\}$ where $uv \notin E$.

Say that a graph H has the **Lee–Yang property** if H *and* all induced subgraphs of H satisfy that the partition function $Z(\boldsymbol{\lambda})$ fails to vanish when each input is in $\overline{\mathbb{D}}$. We show the following now, which is the core of the proof: if H has the Lee–Yang property then so too does any Asano contraction H'. Indeed, if $u, v \in V(H)$ are the vertices to be contracted, consider decomposing

$$Z_{\beta,H}(\boldsymbol{\lambda}) = \lambda_u\lambda_v A_{\beta,H}(\boldsymbol{\lambda}') + \lambda_u B_{\beta,H}(\boldsymbol{\lambda}') + \lambda_v C_{\beta,H}(\boldsymbol{\lambda}') + D_{\beta,H}(\boldsymbol{\lambda}') \tag{1.4}$$

where $\boldsymbol{\lambda}' = (\lambda_w : w \in V \setminus \{u, v\})$. We see then that

$$Z_{\beta,H'}(\lambda, \boldsymbol{\lambda}') = \lambda A_{\beta,H}(\boldsymbol{\lambda}') + D_{\beta,H}(\boldsymbol{\lambda}') \tag{1.5}$$

where we assign the vertex activity λ to the contracted u-v vertex s—we lose the $B_{\beta,H}$ and $C_{\beta,H}$ terms because all terms in H''s partition function correspond to simultaneous lifts to H (i.e. assignments $\sigma(u) = \sigma(v)$), and the $\boldsymbol{\lambda}$ terms in $Z_{\beta,H}$ just count inclusion in the $+$ half. Now, suppose $\lambda_w \notin \overline{\mathbb{D}}$ for $w \in V(H) \setminus \{u, v\}$. We want to show that (1.5) is nonzero in this case, with the knowledge that (1.4) does not vanish for such $\boldsymbol{\lambda}'$. $\boldsymbol{\lambda}'$ is fixed so suppress the function arguments and decorations and let $\lambda_u = \lambda_v = x$, so the quadratic $Ax^2 + (B + C)x + D$ has no roots outside the disk, so the product of its roots lies in the disk: $D/A \in \mathbb{D}$ (by Vieta). But in order for (1.5) to vanish we must have $\lambda = -D/A$, and this we have seen lies in $\mathbb{D}$, which is exactly what we wanted. However, we needed that $A_{\beta,H}(\boldsymbol{\lambda}') \neq 0$, which we recognize as the statement of the Theorem for the induced subgraph $\hat{H} = H[V(H) \setminus \{u, v\}]$ in the following way: $A_{\beta,H}(\boldsymbol{\lambda}') = \beta^{\deg_H u - 1}\beta^{\deg_H v - 1}Z_{\beta,\hat{H}}(\boldsymbol{\lambda}'_A)$ where

$\boldsymbol{\lambda}'_A = \left(\frac{1}{\beta^{\#(N_H(w)\cap\{u,v\})}}\lambda_w : w \in V \setminus \{u,v\}\right)$ where $N_H(w)$ is w's neighborhood in H—the idea is that we want to modify $Z_{\beta,\hat{H}}$ to pick up a β for each neighbor of u and v which is *excluded* from the $+$ half of a given spin assignment.* The Lee–Yang property tells us that indeed $A(\boldsymbol{\lambda}') \neq 0$, using also that if $\lambda_w \notin \overline{\mathbb{D}}$ then so too is $\frac{\lambda_w}{\beta^n}$ for whichever exponent n arises from the neighborhood consideration (importantly $0 < \beta < 1$). However, to fully verify the claim we also need to check the Lee–Yang property for all induced subgraphs H'' of the Asano contraction H'. We readily handle this by the following: either $s \notin V(H'')$, in which case H'' is already an induced subgraph of H, or $s \in V(H'')$ and H'' is an Asano contraction of an induced subgraph of H, namely $\tilde{H} = H[\{u,v\} \sqcup (V(H'') \setminus \{s\})]$, and then in the argument above $\tilde{H}$ plays the role of H and H'' plays the role of H'.

We now prove the Theorem for the graph on one vertex. Here, $Z(\lambda_0) = 1 + \lambda_0$ and the result clearly holds.

We now prove the Theorem for the graph of just a single edge. Here, $Z(\lambda_1, \lambda_2) = 1 + \beta(\lambda_1 + \lambda_2) + \lambda_1\lambda_2$. If Z vanishes then $\lambda_2 = -\frac{1+\beta\lambda_1}{\beta+\lambda_1}$; notice that this is the negative of a Blaschke factor (for $-\beta \in \mathbb{D}$) evaluated at $1/\lambda_1$, and Blaschke factors map $\mathbb{D}$ to itself, so if $\lambda_1 \notin \overline{\mathbb{D}}$ then λ_2 is actually inside $\mathbb{D}$.

We now prove the Theorem for the disjoint union of graphs G_1 and G_2 each individually having the Lee–Yang property. This is immediate because $Z_{\beta,G_1\sqcup G_2} = Z_{\beta,G_1}Z_{\beta,G_2}$.

Finally, we now prove the Theorem for any graph G on m edges. The perfect matching M_m on $2m$ vertices has the Lee–Yang property, and it is a sequence of Asano contractions from M_m to G. Thus G has the Lee–Yang property, and in particular it satisfies the Theorem. ■

1.2 Monomer-dimer model

Definition 1.6 (monomer-dimer model)**.** Consider a graph $G(V,E)$. For a matching M, say that $u(M)$ is the number of unmatched vertices (singletons; **monomers**) in M, i.e. $u(M) = \#V - 2\#E(M)$. Then we give M the weight $\lambda^{u(M)}$ and assign the partition function

$$Z_G(\lambda) = \sum_M \lambda^{u(M)}$$

to normalize the distribution on matchings.

λ represents the tendency for sites to prefer to be unmatched (monomers) or matched (**dimers**); high λ corresponds to a preference for monomers.

*We actually can also see that

$$B_{\beta,H}(\boldsymbol{\lambda}') = \beta^{\deg_H u - 1} Z_{\beta,\hat{H}}(\boldsymbol{\lambda}'_B), \qquad \boldsymbol{\lambda}'_B = \left(\frac{\beta^{\#(N_H(w)\cap\{v\})}}{\beta^{\#(N_H(w)\cap\{u\})}}\lambda_w : w \in V \setminus \{u,v\}\right);$$

$$C_{\beta,H}(\boldsymbol{\lambda}') = \beta^{\deg_H v - 1} Z_{\beta,\hat{H}}(\boldsymbol{\lambda}'_C), \qquad \boldsymbol{\lambda}'_C = \left(\frac{\beta^{\#(N_H(w)\cap\{u\})}}{\beta^{\#(N_H(w)\cap\{v\})}}\lambda_w : w \in V \setminus \{u,v\}\right);$$

$$D_{\beta,H}(\boldsymbol{\lambda}') = Z_{\beta,\hat{H}}(\boldsymbol{\lambda}'_D), \qquad \boldsymbol{\lambda}'_D = \left(\beta^{\#(N_H(w)\cap\{u,v\})}\lambda_w : w \in V \setminus \{u,v\}\right);$$

but these function evaluations are not needed for the proof at hand, nor do they have the same quality of guarantees.

Consider the expected fraction $U(\lambda)$ of monomers, which is

$$U(\lambda) = \frac{\sum_M \frac{u(M)}{\#V} \lambda^{u(M)}}{\sum_M \lambda^{u(M)}} = \frac{1}{\#V Z(\lambda)} \left(z \frac{\partial}{\partial z} Z(z) \right) \bigg|_{z=\lambda} = -\frac{1}{\#V} \left(z \frac{\partial}{\partial z} \log Z(z) \right) \bigg|_{z=\lambda}. \tag{1.7}$$

Theorem 1.8 (Heilmann–Lieb '72)
Let G be a finite graph and $Z = Z_G$ the partition function of the monomer-dimer model. Then, all of Z's zeroes lie on $i\mathbb{R}$.

1.3 Broad-strokes picture

This area of study can be motivated by phase transitions in magnetic materials, where at the "Curie temperature" spontaneous magnetism is lost. One attempt at modeling this (via the ferromagnetic Ising model, §1.1) is to assign a sign (spin) at each site in a lattice, which interact and favor being aligned with their neighbors. This favoring is quantified with weights on entire configurations, with the interactions getting weaker as temperature β increases. There might also be a magnetic field, whose strength separately induces the sites to favor a particular spin. The quantity to study here, **magnetization** $M(\beta)$, is the fraction of +s, so that $M(\beta) = 1/2$ would mean that there is no magnetism. The phase transition should appear as a discontinuity in some derivative of M, however there should be no phase transition in the magnetic field.

Finite graphs give polynomials, which lack discontinuities, so one instead can study limits of larger and larger graphs. For no external magnetic field, Ising showed in 1925 that when the lattice is $\mathbb{Z}$, M remains analytic but Onsager showed in 1944 that when the lattice is $\mathbb{Z}^2$, M'' has a singularity. It was then Lee–Yang in 1952 who showed that mathematically there is no phase transition with respect to the magnetic field, as has been observed experiementaly.

The general situation for a graph G is that the external field arises formally through the parameter λ and the partition function, which normalizes the weighted distribution over physical objects (spins, in the above situation and §1.1; pairings (matchings, monomers/dimers) in §1.2), is denoted $Z_G(\lambda)$. We can also study the related **free energy**

$$F_G(\lambda) = -\frac{1}{\#V(G)} \log Z_G(\lambda),$$

which we see gives interesting formulas for possible observables in (1.2) and (1.7). If F_G is analytic on some domain $\Omega \subset \mathbb{C}$ in either setting, or any analogous setting for a different model, then there cannot be a phase transition inside Ω; the domain of particular interest is $\mathbb{R}_+$ for the above settings.

One setting with a large body of work is approximating $G = \mathbb{Z}^2$ (infinite; might have singularities in F_G) by squares G_n (finite; cannot have singularities in F_{G_n}). Onsager's work referenced earlier was with $\lambda = 1$ and β the parameter exhibiting a singularity. Yang–Lee* showed the following:

*In a different paper from that which contains **Theorem 1.3**.

Theorem 1.9 (Yang–Lee '52)
Let $F_G(z) = \lim_{n\to\infty} F_{G_n}(z)$. If Z_{G_n} lacks roots in the open domain $\Omega \subset \mathbb{C}$ then F_G is analytic in Ω, so that if $I \subset \mathbb{R} \cap \Omega$ is an interval then there are no phase transitions on I.

This is perhaps intuitively true but the analytic details are the content of the theorem.

2 The Heisenberg XXZ model

(Following a paper of Asano)

Let X_i, Y_i, Z_i be the respective Pauli matrices acting on the ith site (and as identity elsewhere).

Definition 2.1 (Heisenberg XXZ model)**.** Pick $n \in \mathbb{N}$. Consider the operators

$$H_{i,j} = \frac{1}{2}\big(Z_i Z_j - \mathrm{id} + \gamma_{i,j}(X_i X_j + Y_i Y_j)\big)$$

for $-1 < \gamma_{i,j} < 1$ symmetric, and the Hamiltonian

$$H = -\sum_{i<j} J_{i,j} H_{i,j}$$

for $J_{i,j} > 0$ symmetric, all indices between 1 and n.

We consider the **magnetization** $M = \sum_i Z_i$ and magnetic field h. M actually commutes with H; clearly this is the case with the $Z_i Z_j$ and id terms, and the nontrivial commutativity comes from the X's and Y's "switching places" as applicable (and using that $Y_i Y_j$ is a real matrix). It is also not hard to check that

$$\langle \neg\sigma | H_{i,j} | \neg\sigma \rangle = \langle \sigma | H_{i,j} | \sigma \rangle. \tag{2.2}$$

We are interested in the **partition function**

$$Q(z) = \operatorname{tr} \exp(\beta(hM - H)) \sum_{\sigma \in \{\pm 1\}^n} \langle \sigma | z^M e^{-\beta H} | \sigma \rangle$$

where z takes the role $e^{\beta h}$ for inverse temperature $\beta = \frac{1}{kT}$, and each entry in σ corresponds to a different qubit/site/wire, and $\{|\sigma\rangle : \sigma \in \{\pm\}^n\}$ forming a basis for the qubits, with $\sigma_i = \pm 1$ corresponding to the ± 1-eigenfunction of Z_i.

Consider the function

$$\Phi(\boldsymbol{z}) = \sum_\sigma \boldsymbol{z}^\sigma \langle \sigma | e^{-\beta H} | \sigma \rangle$$

where $\boldsymbol{z} = \boldsymbol{z}_n = (z_i : 1 \leqslant i \leqslant n)$ (the decoration will be dropped if context is clear), and $\boldsymbol{z}^\sigma = \prod_i z_i^{\sigma_i}$. For the Ising model, it is known that $\Phi(\boldsymbol{z}) \neq 0$ when for all i, $z_i \notin \mathbb{D}$ and there exists j for which $z \notin \overline{\mathbb{D}}$; this is the **Lee–Yang lemma**. *An analagous result will* not *be proved here.* Instead, we get at $Q(z)$ another way. Consider the following terminology:

Definition 2.3 (Lee–Yang lemma)**.** We say that a function $f(\boldsymbol{z})$ **satisfies the Lee–Yang lemma** if whenever for all i, $z_i \notin \mathbb{D}$ and there exists j for which $z \notin \overline{\mathbb{D}}$, it is the case that $f(\boldsymbol{z}) \neq 0$.

We will construct **perturbation series** $\Phi_N(\boldsymbol{z})$ and prove the Lee–Yang lemma about these functions. They are

$$\Phi_N(\boldsymbol{z}) = \sum_{\sigma} \boldsymbol{z}^{\sigma} \langle \sigma | \Bigg(\underbrace{\prod_{i>j} \exp\left(\frac{\beta J_{i,j}}{N} H_{i,j} \right)}_{P(N)} \Bigg)^N |\sigma\rangle$$

and we recognize that $P(N)^N \approx e^{-\beta H}$ for large N, by Trotterization, and that moreover the order of the product is inessential (we will pick an order later on). This function obeys

$$\Phi_N(\boldsymbol{z}) = \Phi_N\left(\boldsymbol{z}^{-1}\right) \tag{2.4}$$

(inverse taken entrywise) due to inversion-symmetry of spins (namely, (2.2)) so that $(\boldsymbol{z}^{-1})^{\sigma} = \boldsymbol{z}^{\neg\sigma}$ and the coefficient of $\boldsymbol{z}^{\neg\sigma}$ is the same as that of $\boldsymbol{z}^{\sigma}$.

Another perturbation series is for Q:

$$Q_N(z) = \sum_{\sigma \in \{\pm 1\}^n} \langle \sigma | z^M P(N)^N | \sigma \rangle$$

and it follows that

$$Q_N(z) = \Phi_N(z\mathbf{1}) \tag{2.5}$$

(i.e. $z\mathbf{1}$ is constant z-valued). We thus have a conditional result, combining (2.5) with (2.4), that once Φ_N is known to satisfy the Lee–Yang lemma, all of Φ_N's roots must lie on S^1. Then, $z^n Q_N(z)$ is a degree-$2n$ polynomial with coefficients converging to those of $z^n Q(z)$ in N. Since roots are continuous in the coefficients, it follows in turn that Q's roots would all lie on S^1. Thus, we must prove the following:

Theorem 2.6 (T.1)

Φ_N satisfies the Lee–Yang lemma for all $N \in \mathbb{N}$.

Consider the function

$$F(\boldsymbol{z}, \boldsymbol{\zeta}) = \sum_{\sigma} \sum_{\sigma'} \boldsymbol{z}^{\sigma} \langle \sigma | e^{-\beta H} | \sigma' \rangle \boldsymbol{\zeta}^{\sigma'}$$

where we are weighting each entry in $e^{-\beta H}$ with separate weights for the 'row' and 'column.' We will relate it to Φ later. For now, we construct an analogous pertutbation series

$$F_N(\boldsymbol{z}, \boldsymbol{\zeta}) = \sum_{\sigma} \sum_{\sigma'} \boldsymbol{z}^{\sigma} \langle \sigma | P(N)^N | \sigma' \rangle \boldsymbol{\zeta}^{\sigma'}.$$

The plan will be to prove that F_N satisfying the Lee–Yang lemma will imply that Φ_N does too (this is the first step of §2.4). We actually will do this using the generalized

$$F_{N,N'}(\boldsymbol{z}, \boldsymbol{\zeta}) = \sum_{\sigma} \sum_{\sigma'} \boldsymbol{z}^{\sigma} \langle \sigma | P(N')^N | \sigma' \rangle \boldsymbol{\zeta}^{\sigma'}, \tag{2.7}$$

and the goal becomes that

Theorem 2.8 (T.2)
$F_{N,N'}$ satisfies the Lee–Yang lemma for all $N, N' \in \mathbb{N}$.

This decoupling between N and N' allows the simplification of considering instead of $J_{i,j}$ just the values $J'_{i,j} = \frac{J}{N'}$ and thus $N' = 1$.

We move now to general functions which will satisfy the Lee–Yang lemma (**Lee–Yang functions**).

2.1 Lee–Yang functions

Definition 2.9 (D.1 (I)). Call $f(\boldsymbol{z})$ **rationally multiaffine** if $\boldsymbol{z}^{\mathbf{1}} f(\boldsymbol{z})$ is multiaffine in each variable z_i^2 and has nonzero degree-$2n$ coefficient (i.e. of $\boldsymbol{z}^{2\mathbf{1}}$).

That is, the structure of such functions is $\sum_{\sigma} \alpha_\sigma \boldsymbol{z}^\sigma$ for some coefficients α_σ with $\alpha_{\mathbf{1}} \neq 0$.

Definition 2.10 (D.1). We say that $f(\boldsymbol{z}) \in L'(\boldsymbol{z})$ if it is rationally multiaffine, and that $f(\boldsymbol{z}) \neq 0$ whenever $z_i \notin \mathbb{D}$ for all i.

This provides further constraints on the α_σs.

L' obeys

Proposition 2.11 (T.3)
For any $f(\boldsymbol{z}) \in L'(\boldsymbol{z})$ and i, $f = z_i g_i(\boldsymbol{z}_{\neg i}) - \frac{1}{z_i} h_i(\boldsymbol{z}_{\neg i})$ where $g_i(\boldsymbol{z}_{\neg i}), h_i(\boldsymbol{z}_{\neg i}) \in L'(\boldsymbol{z}_{\neg i})$.

We use L' to build the following:

Definition 2.12 (D.2). We say that $f(\boldsymbol{z}) \in L(\boldsymbol{z})$ if it is rationally multiaffine, and that for any i, $f = g\left(z_i - \frac{a}{z_i}\right)$ where $g(\boldsymbol{z}_{\neg i}) \in L'(\boldsymbol{z}_{\neg i})$ and a is a function of $\boldsymbol{z}_{\neg i}$ where if $z_j \notin \mathbb{D}$ for all $j \neq i$ then $a(\boldsymbol{z}_{\neg i}) \in \overline{\mathbb{D}}$.

Notice that the only difference between **Definition 2.12** and the second half of **Proposition 2.11** is whether a lands in $\overline{\mathbb{D}}$ or $\mathbb{D}$ (resp.). This definition is remarkably powerful:

Theorem 2.13 (T.4)
Suppose f is rationally multiaffine. Then $f(\boldsymbol{z}) \in L(\boldsymbol{z})$ if and only if f satisfies the Lee–Yang lemma (in the sense of **Definition 2.3**).

Thus, we can rightfully think of L as the **Lee–Yang functions**.

(We omit the proofs of **Proposition 2.11** and **Theorem 2.13** since they are not essential to the main line of reasoning.)

2.2 Two examples of Lee–Yang functions

Consider $n = 1$ and $N = 1$. Then, we have $F_1(z, \zeta) = z\zeta + \frac{1}{z\zeta}$.

Lemma 2.14

$F_1(z, \zeta)$ is a Lee–Yang function.

Proof. Use the substitution $\tau = z\zeta$, so $F_1(z, \zeta) = F_1(\tau) = \tau + \frac{1}{\tau} = \tau + \frac{\overline{\tau}}{|\tau|^2}$. If $F_1(\tau) = 0$ then looking at imaginary parts, we have $\operatorname{Im}\tau = \frac{\operatorname{Im}\tau}{|\tau|^2}$, so $|\tau| = 1$. However, the Lee–Yang lemma insists that F_1 not vanish when both z and ζ lie outside the disk, with one lying outside the closed disk, so $|\tau| = |z\zeta| > 1$, a violation of the necessary condition just derived for F_1 to vanish. Thus indeed F_1 fails to vanish on such (z, ζ). ∎

Consider now $n = 2$. Then, we have

$$F_1(z_1, z_2, \zeta_1, \zeta_2) = \sum_{\sigma,\sigma'} z_1^{\sigma_1} z_2^{\sigma_2} \langle\sigma| \exp(K_{1,2}H_{1,2})|\sigma'\rangle \zeta_1^{\sigma_1'} \zeta_2^{\sigma_2'}$$

(for appropriate $K = K_{1,2}$). Notice that the Hamiltonian considerations have simplified massively in light of the few qubits.

Proposition 2.15 (Appendix 2)

$F_1(z_1, z_2, \zeta_1, \zeta_2)$ is a Lee–Yang function.

Proof. Call $\gamma = \gamma_{1,2}$. Then, one may calculate—this is where we most directly use the structure of the Hamiltonian—that

$$F_1(z_1, z_2, \zeta_1, \zeta_2) = z_1 z_2 \zeta_1 \zeta_2 + \frac{1}{z_1 z_2 \zeta_1 \zeta_2} + e^{-K}\cosh(\gamma K)\left(\frac{z_1\zeta_1}{z_2\zeta_2} + \frac{z_2\zeta_2}{z_1\zeta_1}\right) + e^{-K}\sinh(\gamma K)\left(\frac{z_1\zeta_2}{z_2\zeta_1} + \frac{z_2\zeta_1}{z_1\zeta_2}\right). \tag{2.16}$$

We isolate the coefficient of z_1, which is

$$z_2\zeta_1\zeta_2\left(1 + \frac{e^{-K}}{z_2^2}\left(\frac{\cosh(\gamma K)}{\zeta_2^2} + \frac{\sinh(\gamma K)}{\zeta_1^2}\right)\right).$$

This function lies in $L(z_2, \zeta_1, \zeta_2)$ since for real x, $|\cosh x| + |\sinh x| = e^{|x|}$ so we use the triangle inequality and the fact that $|\gamma K| < K$. Clearly (by symmetry) an identical argument suffices for the other variables.

Suppose now towards contradiction that F_1 vanishes where (WLOG) $z_1 \notin \overline{\mathbb{D}}$ and $z_2, \zeta_1, \zeta_2 \notin \mathbb{D}$. We shall show that there exist $z_1' \notin \overline{\mathbb{D}}$ and $z_2' \in S^1$ with $F_1(z_1', z_2', \zeta_1, \zeta_2) = 0$. Write $F_1(z_1, z_2, \zeta_1, \zeta_2) = z_2 A(z_1, \zeta_1, g z_2) + z_2^{-1} B(z_1, \zeta_1, \zeta_2)$ (A and B are readily retrievable from (2.16)) and in the zero locus view z_2 as a function of z_1, namely $\sqrt{-\frac{B(z_1,\zeta_1,\zeta_2)}{A(z_1,\zeta_1,\zeta_2)}}$. In the limit $|z_1| \longrightarrow \infty$, z_2 limits to $\frac{ie^{-\frac{K}{2}}}{\zeta_1\zeta_2}\sqrt{\cosh(\gamma K)\zeta_1^2 + \sinh(\gamma K)\zeta_2^2}$, which is clearly of magnitude smaller than 1. Thus there is some (z_1', z_2') attained on the contour taking z_1 to ∞ which attains $|z_2'| = 1$.

The above argument has nothing special to do with z_2, i.e. we can find $\zeta_1', \zeta_2' \in S^1$ with $F_1(z_1''', z_2', \zeta_1', \zeta_2') = 0$. Rename back to z_1, z_2, ζ_1, ζ_2, resp. We now solve for z_1 in terms of these other values in S^1:

$$z_1 = -\frac{1 + e^{-K} z_2^2 \big(\cosh(\gamma K)\zeta_2^2 + \sinh(\gamma K)\zeta_1^2\big)}{1 + e^{-K}\overline{z_2}^2 \Big(\cosh(\gamma K)\overline{\zeta_2}^2 + \sinh(\gamma K)\overline{\zeta_1}^2\Big)} \overline{z_2}^2 \overline{\zeta_1}^{-2} \overline{\zeta_2}^{-2}$$
$$= -\frac{\overline{z_2}^2 + e^{-K} \big(\cosh(\gamma K)\zeta_2^2 + \sinh(\gamma K)\zeta_1^2\big)}{1 + \overline{z_2}^2 e^{-K} \Big(\cosh(\gamma K)\overline{\zeta_2}^2 + \sinh(\gamma K)\overline{\zeta_1}^2\Big)} \overline{\zeta_1}^{-2} \overline{\zeta_2}^{-2}$$

and this is the negative of a Blaschke factor for $-e^{-K}\big(\cosh(\gamma K)\zeta_2^2 + \sinh(\gamma K)\zeta_1^2\big)$ (which we know to lie in $\mathbb{D}$), evaluated at $\overline{z_2}^2$, times $-\overline{\zeta_1}^{-2}\overline{\zeta_2}^{-2} \in S^1$. As the Blaschke factors map the closed disk to itself, this contradicts that $z_1 \notin \overline{\mathbb{D}}$. Thus we are done. ∎

2.3 Operations on Lee–Yang functions

Definition 2.17 (derivative operators, D.3)**.** For a function f of n complex variables $\boldsymbol{z}$ and j, k indices, write

$$\begin{aligned} d_j f(\boldsymbol{z}) &= z_j \frac{\partial f}{\partial z_j}(\boldsymbol{z}) \\ I_j f(\boldsymbol{z}) &= -i f(\dots, z_{j-1}, i z_j, z_{j+1}, \dots) \\ D'[z_j, z_k][f(\boldsymbol{z})] &= d_j f(\dots, z_{j-1}, \sqrt{z_j}, z_{j+1}, \dots, z_{k-1}, \sqrt{z_j}, z_{k+1}, \dots) \\ D[z_j, z_k][f(\boldsymbol{z})] &= I_j D'[z_j, z_k][f(\boldsymbol{z})]. \end{aligned}$$

We will only ever use D, and these operators are set up in this way to ensure that the rationally multiaffine structure is maintained. While the meaning is somewhat opaque directly from the definitions, the following calculation is quite enlightening. Write

$$f(\boldsymbol{z}) = z_1 z_2 A_f(\boldsymbol{z}') + \frac{z_1}{z_2} B_f(\boldsymbol{z}') + \frac{z_2}{z_1} C_f(\boldsymbol{z}') + \frac{1}{z_1 z_2} D_f(\boldsymbol{z}') \tag{2.18}$$

where $\boldsymbol{z}' = (z_i : i > 2)$. Then, dropping arguments and decorations,

$$g(z_1, \boldsymbol{z}') = f\big(\sqrt{z_1}, \sqrt{z_1}, \boldsymbol{z}'\big) = z_1 A + B + C + \frac{1}{z_1} D$$

and thus

$$\begin{aligned} d_1 g(z_1, \boldsymbol{z}') &= z_1 A - \frac{1}{z_1} D \\ D[z_1, z_2][f(\boldsymbol{z})] &= z_1 A + \frac{1}{z_1} D. \end{aligned} \tag{2.19}$$

An identical calculation reveals that if

$$f(\boldsymbol{z}) = z_1 A_f(\boldsymbol{z}') + \frac{1}{z_1} B_f(\boldsymbol{z}'), \qquad g(\boldsymbol{\zeta}) = \zeta_1 C_g(\boldsymbol{\zeta}') + \frac{1}{\zeta_1} D_g(\boldsymbol{z}') \tag{2.20}$$

(where here $\boldsymbol{z}' = (z_i : i > 1)$ and $\boldsymbol{\zeta}' = (\zeta_i : i > 1)$) then (again dropping arguments and decorations)

$$D[z_1, \zeta_1][f(\boldsymbol{z}) g(\boldsymbol{\zeta})] = z_1 AC + \frac{1}{z_1} BD. \tag{2.21}$$

Definition 2.22 (Lee–Yang product, D.4)**.** Consider $f(\boldsymbol{z}, \boldsymbol{\zeta})$ and $g(\boldsymbol{z}, \boldsymbol{\sigma})$ rationally multiaffine, decomposed as

$$f(\boldsymbol{z}, \boldsymbol{\zeta}) = \sum_{\sigma,\sigma'} \boldsymbol{z}^{\sigma} a(\sigma, \sigma') \boldsymbol{\zeta}^{\sigma'}$$
$$g(\boldsymbol{z}, \boldsymbol{\zeta}) = \sum_{\sigma,\sigma'} \boldsymbol{z}^{\sigma} b(\sigma, \sigma') \boldsymbol{\zeta}^{\sigma'}$$

for the appropriate coefficient functions a and b. Then, the **Lee–Yang product** is as follows:

$$(f \star g)(\boldsymbol{z}, \boldsymbol{\zeta}) = \sum_{\sigma,\sigma',\sigma''} \boldsymbol{z}^{\sigma} a(\sigma, \sigma') b(\sigma', \sigma'') \boldsymbol{\zeta}^{\sigma''}.$$

Definition 2.23 (Lee–Yang projection, D.4)**.** Consider $f(\boldsymbol{z}, \boldsymbol{\zeta})$ rationally multiaffine, decomposed as

$$f(\boldsymbol{z}, \boldsymbol{\zeta}) = \sum_{\sigma,\sigma'} \boldsymbol{z}^{\sigma} a(\sigma, \sigma') \boldsymbol{\zeta}^{\sigma'}$$

for the appropriate coefficient function a. Then, the **Lee–Yang projection** is as follows:

$$f_{\star}(\boldsymbol{z}) = \sum_{\sigma} \boldsymbol{z}^{\sigma} a(\sigma, \sigma).$$

These operations preserve "Lee–Yang-ness" via **Proposition 2.25**.

2.4 Assembling the perturbation series

In this section, we do the work towards the main results of the paper and defer the technical points about Lee–Yang functions to §2.5. (This section will not be referenced there, so there is no concern of circularity.)

We prove **Theorem 2.6** from **Theorem 2.8**: notice that $\Phi_N(\boldsymbol{z}) = (F_N)_{\star}(\boldsymbol{z}, \boldsymbol{\zeta}) = (F_{N,N})_{\star}(\boldsymbol{z}, \boldsymbol{\zeta})$, so if $F_{N,N}$ is a Lee–Yang function then by **Proposition 2.25** we are done. We prove that hypothesis now, as **Theorem 2.8**.

We proceed with inductions on n and N.

First, to induct on N, we write (from expanding a matrix multiplication in (2.7))

$$F_N(\boldsymbol{z}, \boldsymbol{\zeta}) = \sum_{\sigma,\sigma',\sigma''} \boldsymbol{z}^{\sigma} \langle \sigma | P(1)^{N-1} | \sigma' \rangle \langle \sigma' | P(1) | \sigma'' \rangle \boldsymbol{\zeta}^{\sigma''} = (F_{N-1} \star F_1)(\boldsymbol{z}, \boldsymbol{\zeta}).$$

The induction hypothesis tells us that $F_{N-1}(\boldsymbol{z}, \boldsymbol{\zeta}), F_1(\boldsymbol{z}, \boldsymbol{\zeta}) \in L(\boldsymbol{z}, \boldsymbol{\zeta})$, and **Proposition 2.25** tells us that so too does $F_N(\boldsymbol{z}, \boldsymbol{\zeta})$.

We now enable this induction on N by proving the base case, $N = 1$, via induction on n. It is here that we will multiply the exponential Trotter terms in our special order: reverse-lexicographic on the first entry followed by lexicographic on the second. e.g., if $n = 4$, then the order of pairs is $(3,4), (2,3), (2,4), (1,2), (1,3), (1,4)$. We write f_n for F_1 on n spins. We use the intermediary object

$$G_n(\boldsymbol{z}, \boldsymbol{\zeta}, \boldsymbol{K}_1) = \sum_{\sigma,\sigma'} \boldsymbol{z}^{\sigma} \langle \sigma | \left(\prod_{j} \exp(K_{1,j} H_{1,j}) \right) | \sigma' \rangle \boldsymbol{\zeta}^{\sigma'}$$

where $\boldsymbol{K}_i = (K_{i,j} : j \neq i)$ and we enforce that the product is taken in lexicographic order. From G_n we will construct f_n, but first we need to know that $G_n(\boldsymbol{z}, \boldsymbol{\zeta}, \boldsymbol{K}_1) \in L(\boldsymbol{z}, \boldsymbol{\zeta})$ (where $\boldsymbol{K}_1$ is held fixed). The cases $n = 1$ and $n = 2$ are respectively handled by **Lemma 2.14** and **Proposition 2.15**. Then, for $n > 2$, define

$$G_n(\boldsymbol{z}, \boldsymbol{\zeta}, \boldsymbol{K}_1) = D[u, v][G_{n-1}(\boldsymbol{z}, u, (\zeta_i : i > 1), \boldsymbol{K}_1) G_2(v, z_n, \zeta_1, \zeta_n, K_{1,n})]|_{u=1}$$

(recall D from **Definition 2.17**) and these are all in $L(\boldsymbol{z})$, by **Proposition 2.24**. We then construct

$$f_n(\boldsymbol{z}, \boldsymbol{\zeta}) = \left(\prod_{i=2}^{n} D[u_i, v_i]\right)[f_{n-1}((z_j : j > 1), (u_j : j > 1), (K_{j,k} : 1 < j < k)) G_n(z_1, (v_j : j > 1), \boldsymbol{\zeta}, \boldsymbol{K}_1)]|_{\boldsymbol{u}=\boldsymbol{1}};$$

from **Proposition 2.24**, the techniques of the proof of **Proposition 2.25**, our knowledge of G_n, and the induction hypothesis for f_{n-1}, we conclude that $f_n(\boldsymbol{z}, \boldsymbol{\zeta}) \in L(\boldsymbol{z}, \boldsymbol{\zeta})$. The base case ($f_1$) is once more **Lemma 2.14**. Thus the proof is complete.

2.5 Facts about operations on Lee–Yang functions

Proposition 2.24 (T.7, T.8)

For $f(\boldsymbol{z}) \in L(\boldsymbol{z})$, $D[z_1, z_2][f(\boldsymbol{z})] \in L(\boldsymbol{z})$, and by symmetry this extends to all pairs of indices. For $f(\boldsymbol{z}) \in L(\boldsymbol{z})$ and $g(\boldsymbol{\zeta}) \in L(\boldsymbol{\zeta})$, $D[z_1, \zeta_1][f(\boldsymbol{z})g(\boldsymbol{\zeta})] \in L(\boldsymbol{z}, \boldsymbol{\zeta})$, and again by symmetry this extends to all pairs.

Proof. For the first, we use (2.18) and (2.19). Specialize (2.18) to $z_2 = z_1$ and suppose the function vanishes; using Vieta's formulas for the product of the roots, we find that (2.19) cannot vanish unless $z_1 \in \overline{\mathbb{D}}$. It is essential here that A not vanish, which is given by f's Lee–Yang-ness.

Similarly, for the second, we use (2.20) and (2.21). Specialize (2.20) to $\zeta_1 = z_1$:

$$z_1^2 AC + AD + BC + \frac{1}{z_1^2} BD$$

and suppose the function vanishes; using Vieta's formulas for the product of the roots, we find that (2.21) cannot vanish unless $z_1 \in \overline{\mathbb{D}}$. It is essential here that A and C not vanish, which is given by f and g's Lee–Yang-ness. ■

Notice that this is identical to the strategy used in the proof of **Theorem 1.3**.

Proposition 2.25 (T.11)

For Lee–Yang functions f and g, $f \star g$ and $f_\star$ are also Lee–Yang functions; i.e., if $f(\boldsymbol{z}, \boldsymbol{\zeta}), g(\boldsymbol{z}, \boldsymbol{\zeta}) \in L(\boldsymbol{z}, \boldsymbol{\zeta})$, then $(f \star g)(\boldsymbol{z}, \boldsymbol{\zeta}) \in L(\boldsymbol{z}, \boldsymbol{\zeta})$ and $f_\star(\boldsymbol{z}) \in L(\boldsymbol{z})$.

Proof. For the first, consider $D[u_i, v_i][f(\boldsymbol{z}, \boldsymbol{u})g(\boldsymbol{v}, \boldsymbol{\zeta})] \in L(\boldsymbol{z}, \boldsymbol{u}, (v_j : j \neq i), \boldsymbol{\zeta})$ by **Proposition 2.24**, hence

$$\left(\prod_i D[u_i, v_i]\right)[f(\boldsymbol{z}, \boldsymbol{u})g(\boldsymbol{v}, \boldsymbol{\zeta})] \in L(\boldsymbol{z}, \boldsymbol{u}, \boldsymbol{\zeta})$$

and specializing to $\boldsymbol{u} = \mathbf{1}$ recovers $f \star g$. The easiest way to see this is by analyzing $D[u_1, v_1][f(\boldsymbol{z}, \boldsymbol{u})g(\boldsymbol{v}, \boldsymbol{\zeta})]$ where

$$f(\boldsymbol{z}, \boldsymbol{u})g(\boldsymbol{v}, \boldsymbol{\zeta}) = u_1 v_1 A_f(\boldsymbol{z}, \boldsymbol{u}')C_g(\boldsymbol{v}', \boldsymbol{\zeta}) + \frac{u_1}{v_1} AD_g(\boldsymbol{v}', \boldsymbol{\zeta}) + \frac{v_1}{u_1} B_f(\boldsymbol{z}, \boldsymbol{u}')C + \frac{1}{u_1 v_1} BD$$

and we have $\boldsymbol{u}' = (u_j : j > 1)$, $\boldsymbol{v}' = (v_j : j > 1)$, A_f, B_f, C_g, D_g are defined in analogy to (2.20), and we drop decorations/arguments midway for legibility. Then unsurprisingly

$$D[u_1, v_1][f(\boldsymbol{z}, \boldsymbol{u})g(\boldsymbol{v}, \boldsymbol{\zeta})]|_{u_1 = v_1 = 1} = AC + BD,$$

i.e. we have "glued" the parts where u_1 and v_1 are in the numerator and where they are in the denominator. We then carry this through for all i.

Similarly, for the second, by the same argument $D[z_i, \zeta_i][f(\boldsymbol{z}, \boldsymbol{\zeta})] \in L(\boldsymbol{z})$ and we recognize

$$f_\star = \left(\prod_i D[z_i, \zeta_i]\right)[f(\boldsymbol{z}, \boldsymbol{\zeta})],$$

the quick idea being that in

$$f(\boldsymbol{z}, \boldsymbol{\zeta}) = \sum_{\sigma, \sigma'} \boldsymbol{z}^\sigma a(\sigma, \sigma') \boldsymbol{\zeta}^{\sigma'}$$

when we substitute $\sqrt{z_1}$ for z_1 and ζ_1, there is cancellation precisely when $\sigma_1 \neq \sigma_1'$ and so the z_1-derivative causes such terms to vanish; this persists for each i after 1. ■