Nonbipartite LPS Ramanujan graphs

Zachary Stier

October 2022

In this note, I will present Lubotzky–Phillips–Sarnak's construction of nonbipartite Ramanujan graphs.

1 Preliminaries and notation

Throughout, let p and q be distinct odd primes, both congruent to 1 modulo 4, satisfying $\left(\frac{p}{q}\right) = 1$. For a graph G, write A_G for its unnormalized adjacency matrix.

1.1 Spectral preliminaries

Notation (graph spectrum)**.** For a connected d-regular graph G, let $\sigma(G)$ denote the spectrum of A_G. Say its elements are $d = \lambda_1 > \lambda_2 \geqslant \cdots \geqslant \lambda_n$.

> **Definition** (bulk of spectrum)**.** For a d-regular graph G, the *bulk* of its spectrum is $\sigma(G) \cap \left[-2\sqrt{d-1}, 2\sqrt{d-1}\right]$.

Notation (second absolute eigenvalue)**.** For a graph G, let $\lambda_\star(G)$ denote the largest between the second eigenvalue and negative of the least eigenvalue of A_G, i.e. $\max\{\lambda_2, -\lambda_n\}$.

> **Definition** (Ramanujan graph)**.** A d-regular graph G is *Ramanujan* if $\lambda_\star(G) \leqslant 2\sqrt{d-1}$. That is, the entire spectrum aside from d lies in the bulk.

1.2 Chebyshev polynomials

Notation (Chebyshev polynomials)**.** Let $T_n(x)$ denote the *nth Chebyshev polynomial of the first kind*, and $U_n(x)$ denote the *nth Chebyshev polynomial of the second kind*. We use the convention that $T_n = T_{-n}$ and $U_n = U_{-n}$.

Proposition 1
The following are equivalent formulations of U_n.

(a) $\frac{\sin((n+1)\arccos x)}{\sin \arccos x}$ is a degree-n polynomial in x on $[-1, 1]$, and extends to a polynomial U_n in x on all of $\mathbb{R}$.

(b) Fix $c > 0$. $c^{n/2} U_n\left(\frac{x}{2\sqrt{c}}\right) =: \tilde{U}_n^{(c)}(x)$ is the solution to the recursion

$$H_0^{(c)}(x) = 1, \quad H_1^{(c)}(x) = x, \quad H_n^{(c)}(x) = xH_{n-1}^{(c)}(x) - cH_{n-2}^{(c)}(x).$$

(c) $U_n(x) = \sum_{0 \leqslant j \leqslant n} T_{n-2j}(x)$. (We are not interested in T_n aside from its use in **Proposition 2**.)

Proposition 2
We have that for all $x \in \mathbb{R} \setminus [-1, 1]$ and n even, $T_n(x) \geqslant 2x^n$, so that

$$U_n(x) \geqslant 4x^n.$$

This fact is a consequence of e.g. this Math.StackExchange answer and **Proposition 1**(c).

1.3 Graph statistics

Notation (girth)**.** For a graph G, let its girth be $g(G)$.

Proposition 3
For a d-regular graph $G(V, E)$, the number of nonbacktracking paths of length n beginning at x and ending at y is $\tilde{U}_n^{(d-1)}(A_G)(x, y)$.

Proof idea. We note that the recursion from **Proposition 1**(b) inductively counts the desired quantity by subtracting off the paths that backtrack at the very last step (there are $d - 1$ ways to take a nonbacktracking penultimate step and then immediately backtrack it). ∎

1.4 Number-theoretic preliminaries

Definition. For an integral quadratic form Q taking inputs $\boldsymbol{x} \in \mathbb{Z}^n$, for

$m \in \mathbb{Z}$ let $r_Q(m)$ be the number of solutions to $Q(\boldsymbol{x}) = m$.

Theorem 4 (Eichler; Igusa)
Fix Q to be the norm against $1 \oplus (2q)^3 I_3$. There is an absolute constant $c_1 > 0$ such that $C(p^k) := c_1 \frac{p^{k+1}-1}{p-1}$ satisfies, for all $\varepsilon > 0$, the relation

$$r_Q\left(p^k\right) = C\left(p^k\right) + O\left(p^{k(1/2+\varepsilon)}\right)$$

(the big-Oh term being asymptotic in k).

Lemma 5
Let $G : \mathbb{N} \longrightarrow \mathbb{C}$ be periodic with period coprime to p and satisfy

$$\sum_{d \mid p^k} dG(d) = o(p^k),$$

as a function of k. Then in fact G is identically zero.

Proof. Say $s_k := \sum_{d \mid p^k} dG(d)$. Then we see that $G(p^k) = \frac{1}{p^k}(s_k - s_{k-1}) = \frac{s_k}{p^k} - \frac{1}{p} \cdot \frac{s_{k-1}}{p^{k-1}}$; both terms vanish in k, by hypothesis. However, G is periodic; so if $G(p^\ell) \neq 0$, say $\alpha := \left|G(p^\ell)\right| > 0$, then there exists k such that G's period divides $p^k - p^\ell$ and such that $\frac{s_k}{p^k}, \frac{s_{k-1}}{p^{k-1}} < \frac{\alpha}{2}$, a contradiction as $G(p^k) = G(p^\ell)$. Thus indeed $G(p^\ell) = 0$. ∎

2 Construction

We briefly recapitulate the construction of the graphs $X = X^{p,q}$.

Construction of the graph

Let $S \subset \mathbb{H}(\mathbb{Z})$ consist of solutions to $\mathrm{N}(\alpha) = p$ for which $\frac{1}{2}\operatorname{Tr}(\alpha)$ is positive and odd. Let $\iota \in \mathbb{N}$ satisfy $\iota^2 \equiv -1 \pmod{q}$. Consider the map

$$\phi : a + bi + cj + dk \longmapsto \begin{pmatrix} a + b\iota & c + d\iota \\ -c + d\iota & a - b\iota \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{F}_q).$$

Then $X^{p,q}$ is defined as the Cayley graph for $\mathrm{PSL}(2, \mathbb{F}_q)$ with respect to $\phi(S)$. This graph is $(p+1)$-regular, on $n := \frac{1}{2}(q-1)q(q+1)$ vertices.

Useful interm. constructions

The paper covers in full why this construction actually yields a connected, nonbipartite graph—details that we omit here—however it is convenient to discuss some intermediate steps in the realization of the graphs. We say that

$\Lambda'(2)$ consists of $\alpha \in \mathbb{H}(\mathbb{Z})$ for which $\mathrm{N}(\alpha)$ is *some* natural power of p and $\frac{1}{2}\operatorname{Tr}(\alpha)$ is odd. Impose the following equivalence relation on $\Lambda'(2)$: $\alpha \sim \beta$ if α/β, when viewed in $\mathbb{H}(\mathbb{Q})$, is in the center $\mathbb{Q}$, i.e. $\alpha/\beta = \pm p^m$ for $m \in \mathbb{Z}$. We then let $\Lambda(2) := \Lambda'(2)/\sim$, which is free on $S' \subset S/\sim$, defined as ignoring 'inverse-duplicates' in $S/\sim$: if $\alpha \in S$ then $\overline{\alpha} \in S$, and $[\alpha][\overline{\alpha}] = [1]$, so in compiling S we select only (WLOG) $[\alpha]$. By this freeness and the observation that $\#S = p+1$ (by Jacobi's four square theorem), we may identify elements of $\Lambda(2)$ with vertices in the rooted $(p+1)$-regular tree $\mathbb{T}$, where WLOG we map $[1]$ to the root. We let $\Lambda'(2q) \subset \Lambda'(2)$ be those α for which $2q \mid \left(\alpha - \frac{1}{2}\operatorname{Tr}(\alpha)\right)$, i.e. $2q$ divides each component except the $\mathbb{Z}$-component. Then, $\Lambda(2q) := \Lambda'(2q)/\sim$, and as $\Lambda(2q) \triangleleft \Lambda(2)$, we set $X := \Lambda(2)/\Lambda(2q)$.

3 Properties

Theorem 6

X is Ramanujan.

In the following proof, all big-Oh and little-oh asymptotics are with respect to variable k. For families of variables on the same indexing set indicated by a subscript, equations involving these variables without subscripts indicates suppression of the *same* subscript.

Proof. Recall **Proposition 3**: $\tilde{U}_k^{(p)}(A_X) =: \tilde{U}_k(A_X) =: L_k$ tracks the number of nonbacktracking paths of length k in X. We study its trace in two ways:

Two ways of studying n.b. walk counts

- By the graph's transitivity, L_k's diagonal is constant (all closed walks starting from e are in bijection with those from g via the graph automorphism $x \longmapsto gx$). Thus, $\operatorname{Tr} L_k = nL_k(e,e)$. However, a walk from the identity to itself of length t is a nonbacktracking walk in $\mathbb{T}$ from the identified lift of e to *any* element of $\Lambda(2q)$. Thus, the number of such length-k nonbacktracking paths is the number of points at that distance. This is exactly $\frac{1}{2}r_Q(p^k)$, the one-half factor accounting for the equivalence of solutions $\pm \boldsymbol{x}$ satisfying $Q(\boldsymbol{x}) = p^k$ under $\sim$, where Q is as in **Theorem 4**. All told,
$$\operatorname{Tr} L_k = \frac{n}{2} r_Q\left(p^k\right). \tag{1}$$

- We also know that
$$\operatorname{Tr} L_k = \sum_{j\in[n]} \tilde{U}_k^{(p)}(\lambda_j) = p^{k/2} \sum_{j\in[n]} U_k\left(\frac{\lambda_j}{2\sqrt{p}}\right) \tag{2}$$
since A_G is diagonalizable and $\tilde{U}_k$ is just a polynomial.

We combine (1), (2), and **Theorem 4** to find

$$C(p) + O\left(p^{k(1/2+\varepsilon)}\right) = \frac{2}{n} p^{k/2} \sum_{j\in[n]} U_k\left(\frac{\lambda_j}{2\sqrt{p}}\right). \tag{$\spadesuit$}$$

(The following paragraph is taken essentially verbatim from the paper.) $C(p^k)$ is the "singular series" and it comes from the contribution of the Eisenstein series when expressing the "θ-function" $\theta(z) = \sum_{\boldsymbol{\nu}\in\mathbb{Z}^4} \exp(2\pi i Q(\boldsymbol{\nu})z)$ as a combination of Eisenstein series and a cusp form. That is, $C(p^k)$ is the p^kth Fourier coefficient of a combination of the Eisenstein series of weight two for $\Gamma(16q^2)$. From the known Fourier expansions of Eisenstein series one easily shows that C is of the form

C is a sum over divisors

$$C(n) = \sum_{d|n} dF(d)$$

where $F : \mathbb{N} \longrightarrow \mathbb{C}$ is periodic of period $(2q)^2$, so we are left with

$$\sum_{d|p^k} dF(d) + O\left(p^{k(1/2+\varepsilon)}\right) = \frac{2}{n} p^{k/2} \sum_{j\in[n]} U_k\left(\frac{\lambda_j}{2\sqrt{p}}\right). \tag{$\heartsuit$}$$

We turn to the right-hand side. Write

Study Cheb. evals. of eigenvalues

$$t := \frac{\lambda}{2\sqrt{p}} + \sqrt{\frac{\lambda^2}{4p} - 1} \tag{3}$$

(taken to be in the upper half-plane and with a positive square root, where relevant); this t is the solution to

$$\frac{t + 1/t}{2} = \frac{\lambda}{2\sqrt{p}},$$

so that if $t = e^{i\theta}$ then θ can be seen as the "solution" to $\lambda = 2\sqrt{p}\cos\theta$—but we extend cos to $\mathbb{C}$ as $\frac{e^{iz}+e^{-iz}}{2}$. Similarly, we extend sin to $\mathbb{C}$ as $\frac{e^{iz}-e^{-iz}}{2i}$. Interpreting **Proposition 1**(a) in this light we find that

$$U_k\left(\frac{\lambda}{2\sqrt{p}}\right) = \frac{t^{k+1} - t^{-(k+1)}}{t - t^{-1}}.$$

Algebra reveals that t as in (3) satisfies

$$\frac{t^{k+1} + t^{-(k+1)}}{t + t^{-1}} = \frac{\lambda^2}{4p}\left(\frac{\lambda}{2\sqrt{p}}\right)^k \left(1 - \left(1 - \frac{4p}{\lambda^2}\right)^{\left\lfloor \frac{k+1}{2} \right\rfloor - 1}\right) \tag{4}$$

so that when $2\sqrt{p} < |\lambda| < p+1$, (4) decays as $o\left(p^{k/2}\right)$ from the term $\left(\frac{\lambda}{2\sqrt{p}}\right)^k$; when $\lambda = 2\sqrt{p}$, (4) equals 1; when $|\lambda| < 2\sqrt{p}$, (4) decays still as $o\left(p^{k/2}\right)$ from the term $\frac{4p}{\lambda}$; and when $\lambda = p+1$ we directly compute that $t = \sqrt{p}$ so (4) dominates (it is $\Theta(p^{k/2})$). In particular,

$$\frac{2}{n}p^{k/2}\sum_{j\in[n]} U_k\left(\frac{\lambda_j}{2\sqrt{p}}\right) = \frac{2}{n}\frac{p^{k+1}-1}{p-1} + o\left(p^{k/2}\right) = \frac{2}{n}\sum_{d|p^k} d + o\left(p^{k/2}\right),$$

transforming (♥) into

$$\sum_{d|p^k} dF(d) + O\left(p^{k(1/2+\varepsilon)}\right) = \frac{2}{n}\sum_{d|p^k} d + o\left(p^{k/2}\right) \qquad (\blacklozenge)$$

(and the little-oh term on the right is cancelled by the big-Oh term on the left).

Compute $C(p)$ We apply **Lemma 5** to $F(d) - \frac{2}{n}$ in (♦). Thus we have computed c_1 from **Theorem 4** to be $\frac{2}{n}$.

Bound remaining eigenvalues Finally, we show that the remaining eigenvalues are entirely contained within the bulk. We consider (♠), with the newfound knowledge of c_1. When $j = 1$, $\lambda = p+1$ so the corresponding term in the sum equals $\frac{2}{n}\frac{p^{k+1}-1}{p-1}$, so that we are left with

$$\sum_{j>1} U_k\left(\frac{\lambda_j}{2\sqrt{p}}\right) = O\left(p^{k\varepsilon}\right) \qquad (\clubsuit)$$

for any fixed ε. Let k be even. If λ—towards contradiction—is not in the bulk then $\frac{\lambda}{2\sqrt{p}} = \pm(1+\delta)$ for some $\delta > 0$. **Proposition 2** tells us that the corresponding term in the sum above grows faster than $(1+\delta)^k$, and by definite parity of the Chebyshev polynomials—they will all be positive on $\mathbb{R} \setminus [-1,1]$ and bounded by $O(k)$ on $[-1,1]$—we see that

$$(1+\delta)^k < \sum_{j>1} U_k\left(\frac{\lambda_j}{2\sqrt{p}}\right) = O\left(p^{k\varepsilon}\right).$$

The contradiction is evident now: select ε so that $p^{\varepsilon} < 1+\delta$ (say, $\varepsilon = \frac{1}{2}\log_p(1+\delta)$). Then clearly the left-hand side grows faster than the right-hand side, a contradiction. Thus indeed we must have the entire remainder of the spectrum inside the bulk. This is the same as saying that the graph is Ramanujan. ■

Remark. The analysis of

$$\sum_{j\in[n]} U_k\left(\frac{\lambda_j}{2\sqrt{p}}\right)$$

is completed in a different fashion in this MathOverflow answer; in particular the final step of the proof above is substantively different.

Proposition 7
$g(X) \geqslant 2\log_p q$.

Proof. WLOG by transitivity that a shortest cycle under consideration, say of length t, begins and ends at e. This lifts to a nonbacktracking path in $\mathbb{T}$ from $[1]$ to $[\alpha] \in \Lambda(2q)$, $[\alpha]$ at distance t. Take a representative of this class $\alpha = a + 2q(bi + cj + dk)$ (with $\gcd(a, b, c, d) = 1$). Then,

$$p^t = \mathrm{N}(\alpha) = a^2 + 4q^2(b^2 + c^2 + d^2). \tag{5}$$

At least one of $\{b, c, d\}$ is nonzero, otherwise $[\alpha] = [1]$, in particular (WLOG) $b^2 \leqslant 1$, so (5) becomes $p^t \geqslant 4q^2 \geqslant q^2$. The result follows. ■