Number Theory Final Exam, Spring 2014

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1. (2 points) Write down a Weierstrass equation for each of the following elliptic curves:

(1) The cubic curve $X^3 + Y^3 = nZ^3$ in $\mathbb{P}^2_{\mathbb{Q}}$ with the distinguished point $(1, -1, 0)$. Here $n \in \mathbb{Q}^{\times}$.

(2) The normalization of the curve $y^2 = x(x-a)(x-b)(x-c)$ (viewed as a closed curve in $\mathbb{P}^2_{\mathbb{Q}}$ by homogenizing the equation) with the distinguished point $(x, y) = (0, 0)$. Here $a, b, c \in \mathbb{Q}$, and $0, a, b, c$ are distinct.

2. (3 points) Let E be an elliptic curve over $\mathbb{Q}$, and $E^{(d)}$ be the quadratic twist of E by a non-square number $d \in \mathbb{Q}^{\times}$. Prove

$$\begin{aligned} L(E_K, s) &= L(E, s)\ L(E^{(d)}, s), \\ \text{rank } E(K) &= \text{rank } E(\mathbb{Q}) + \text{rank } E^{(d)}(\mathbb{Q}). \end{aligned}$$

Here $K = \mathbb{Q}(\sqrt{d})$ is the quadratic field, and E_K is the base change of E to K. For the quadratic twist, recall that if E has an equation $y^2 = x^3 + a_2x^2 + a_4x + a_6$, then $E^{(d)}$ has an equation $dy^2 = x^3 + a_2x^2 + a_4x + a_6$. (Hint: Consider the action of $\mathrm{Gal}(K/\mathbb{Q})$ on $E(K)$.)

3. (2 points) Let K be a field with $\mathrm{char}(K) \neq 2$. Let E be an elliptic curve over K given by

$$y^2 = x^3 + ax + b, \quad a, b \in K.$$

Let $\omega = \dfrac{dx}{2y}$ be a rational differential on E. Prove that ω is regular on E, and it is translation-invariant in the sense that $\tau_P^* \omega = \omega$ for any $P \in E(\overline{K})$. Here $\tau_P : E_{\overline{K}} \to E_{\overline{K}}$ denotes the translation map by P. (You can finish the problem by either explicit computation or abstract algebraic geometry.)

4. (3 points) Let K be a *local field* with $\mathrm{char}(K) \neq 2$. Denote by O_K the valuation ring, $\wp$ the maximal ideal of O_K, and $k = \mathbb{F}_q$ the residue field. Let E be an elliptic curve over K given by

$$y^2 = x^3 + ax + b, \quad a, b \in O_K.$$

Assume that the discriminant $-16(4a^3 + 27b^2) \in O_K^\times$, so that the equation has good reduction $\overline{E}$ over k. Let $\omega = \dfrac{dx}{2y}$ be the invariant differential on E. Prove

$$\int_{E(K)} |\omega| = q^{-1}|\overline{E}(k)|.$$

(The integration is of Tamagawa type. If you are not familiar with it, here is an explanation. Find a coordinate chart $\{f_\alpha | \alpha \in S\}$ of $E(K)$ in the sense that, for any $\alpha \in S$, f_α is an injective map of the form:

$$f_\alpha : \wp \longrightarrow E(K), \quad t \longmapsto (X_\alpha(t), Y_\alpha(t)), \quad X_\alpha, Y_\alpha \in O_K[[t]].$$

Furthermore, assume that $E(K)$ is the *disjoint* union of $f_\alpha(\wp)$ (when α takes all elements of S). Then define

$$\int_{E(K)} |\omega| := \sum_{\alpha \in S} \int_\wp |f_\alpha^* \omega / dt| dt.$$

Here $f_\alpha^*\omega/dt$ is a function on $\wp$, and $|f_\alpha^*\omega/dt|$ is the absolute value normalized by $|a| = q^{-\mathrm{ord}_\wp(a)}$. The Haar measure dt on $\wp$ is normalized so that $\mathrm{vol}(\wp) = q^{-1}$. The definition is independent of the coordinate chart.)