MATH 1A FINAL EXAM SAMPLE, XINYI YUAN, FALL 2014

1. Find the following limits:

$$\lim_{x\to 1}\frac{x^3-1}{x^2-1}, \qquad \lim_{x\to 0} x^{-2}(e^x+e^{-x}-2).$$

2. Find the derivatives of the following functions:

$$f(x)=\cos(x)\ln\sin(x^2), \qquad g(x)=\int_{x^2}^{x^3}\sin(t^2)dt.$$

3. Compute the following integrals:

$$\int\frac{x^3}{\sqrt{x^2+1}}dx, \qquad \int_1^2\frac{1}{x}(\ln x+x^2)dx.$$

4. Find the maximal value and the minimal value of the function

$$f(x)=xe^{-\frac{x^2}{8}}$$

on the interval $[-1,4]$.

5. Compute the area of the region in the xy-plane bounded by the graphs of

$$y=x, \quad y=1-\frac{1}{2}x, \quad y=\frac{1}{2}x^2,$$

and lying below the lines $y=x$ and $y=1-\frac{1}{2}x$.

6. Let R be the region in the previous problem, i.e., the region in the xy-plane bounded by the graphs of

$$y=x, \quad y=1-\frac{1}{2}x, \quad y=\frac{1}{2}x^2,$$

and lying below the lines $y=x$ and $y=1-\frac{1}{2}x$. Compute the volume of the solid obtained by rotating R about the x-axis.

7. Let S be the region in the xy-plane bounded by the graphs of the equations

$$y=x^3-4x+4, \quad y=0, \quad x=1, \quad x=3.$$

Compute the volume of the solid obtained by rotating S about the y-axis.

#1(a): $\frac{x^3-1}{x^2-1} = \frac{(x-1)(x^2+x+1)}{(x-1)(x+1)} = \frac{x^2+x+1}{x+1} \to \frac{3}{2}$.

Alternative approach: L'Hospital's Rule.

#1(b): Apply L'Hospital's rule twice.

$$\lim_{x\to 0} \frac{e^x+e^{-x}-2}{x^2} = \lim_{x\to 0} \frac{e^x-e^{-x}}{2x} = \lim_{x\to 0} \frac{e^x+e^{-x}}{2} = 1.$$

#2(a):
$$f'(x) = -\sin x \ln \sin(x^2) + \cos x \frac{1}{\sin(x^2)} \cos(x^2)\cdot 2x.$$

#2(b): Let $G(x)$ be an anti-derivative of $\sin(x^2)$.

Then $G'(x) = \sin(x^2)$, and

$$g(x) = G(x^3) - G(x^2).$$

thus
$$g'(x) = G'(x^3)\cdot 3x^2 - G'(x^2)\cdot 2x$$
$$= 3x^2 \sin(x^6) - 2x \sin(x^4).$$

#3(a): Set $u = x^2+1$.

Then $du = 2x\,dx$, $x\,dx = \frac{1}{2}du$.

$$\int \frac{x^3}{\sqrt{x^2+1}}\,dx$$

$$= \int \frac{x^2}{\sqrt{x^2+1}}\,x\,dx$$

$$= \int \frac{u-1}{\sqrt{u}}\cdot\frac{1}{2}\,du = \frac{1}{2}\int (u^{\frac{1}{2}} - u^{-\frac{1}{2}})\,du$$

$$= \frac{1}{3}u^{\frac{3}{2}} - u^{\frac{1}{2}} + C.$$

#3(b): $\int_1^2 \frac{1}{x}(\ln x + x^2)\,dx$

$$= \int_1^2 \frac{1}{x}\ln x\,dx + \int_1^2 x\,dx$$

$$= \int_0^{\ln 2} u\,du + \frac{1}{2}x^2\Big|_1^2$$ (set $u = \ln x$ in the first integral)

$$= \frac{1}{2}(\ln 2)^2 + \frac{3}{2}.$$

#4:

$$f'(x) = e^{-\frac{x^2}{8}} + x\, e^{-\frac{x^2}{8}}\left(-\frac{1}{8}\cdot 2x\right)$$

$$= \left(1 - \frac{1}{4}x^2\right) e^{-\frac{x^2}{8}}$$

Critical points: $f'(x) = 0 \Rightarrow x = \pm 2$.

Compare:

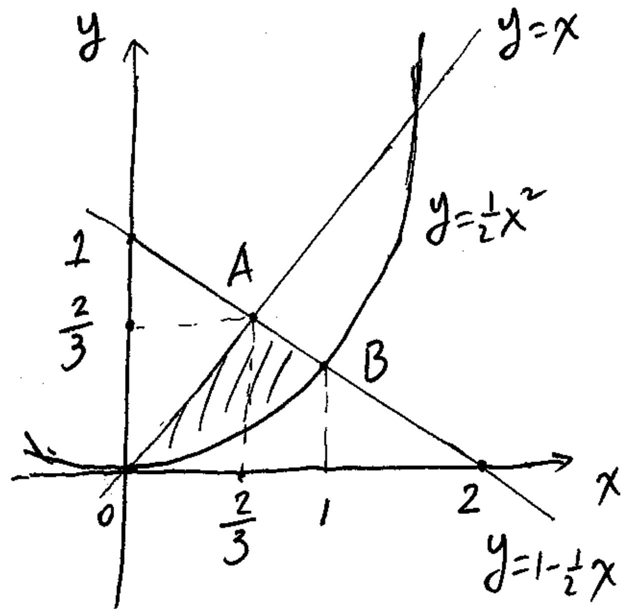
$f(2) = 2e^{-\frac{1}{2}} \rightarrow$ maximal.

$\boxed{\cancel{f(-2) = -2e^{-\frac{1}{2}} \rightarrow \text{minimal}}}$ $\quad -2 \notin [-1,4]$.

$f(-1) = -e^{-\frac{1}{8}} \rightarrow$ minimal

$f(4) = 4e^{-2}$

#5: First, figure out the intersection points.



①. $\begin{cases} y = x \\ y = 1 - \frac{1}{2}x \end{cases}$

$\Rightarrow A\left(\frac{2}{3}, \frac{2}{3}\right)$

②. $\begin{cases} y = \frac{1}{2}x^2 \\ y = 1 - \frac{1}{2}x \end{cases}$

$\Rightarrow B\left(1, \frac{1}{2}\right)$. The equations give 2 solutions, but $\left(1, \frac{1}{2}\right)$ is what we need.

Second, to compute the area, cut the region by the line $x=\frac{2}{3}$. Then

$$A = \int_0^{\frac{2}{3}} \left(x-\frac{1}{2}x^2\right) dx + \int_{\frac{2}{3}}^{1} \left(\left(1-\frac{1}{2}x\right)-\frac{1}{2}x^2\right) dx$$

$$= \left(\frac{1}{2}x^2-\frac{1}{6}x^3\right)\Big|_0^{\frac{2}{3}} + \left(x-\frac{1}{4}x^2-\frac{1}{6}x^3\right)\Big|_{\frac{2}{3}}^{1}$$

$$= \frac{1}{4}.$$

#6: Integrate along x-axis.

The section $A(x)$ is a washer everywhere.

Still cut the region by the line $x=\frac{2}{3}$.

Get

$$V = \int_0^{\frac{2}{3}} \pi\left(x^2-\left(\frac{1}{2}x^2\right)^2\right) dx + \int_{\frac{2}{3}}^{1} \pi\left(\left(1-\frac{1}{2}x\right)^2-\left(\frac{1}{2}x^2\right)^2\right) dx$$

$$= \pi \int_0^{\frac{2}{3}} \left(x^2-\frac{1}{4}x^4\right) dx + \pi \int_{\frac{2}{3}}^{1} \left(1-x+\frac{1}{4}x^2-\frac{1}{4}x^4\right) dx$$

$$= \pi\left(\frac{1}{3}x^3-\frac{1}{20}x^5\right)\Big|_0^{\frac{2}{3}} + \pi\left(x-\frac{1}{2}x^2+\frac{1}{12}x^3-\frac{1}{20}x^5\right)\Big|_{\frac{2}{3}}^{1}$$

$$= \frac{22}{135}\pi.$$

#7: First, check

$x^3-4x+4 \geq 0$

for any $x \in [1,3]$.

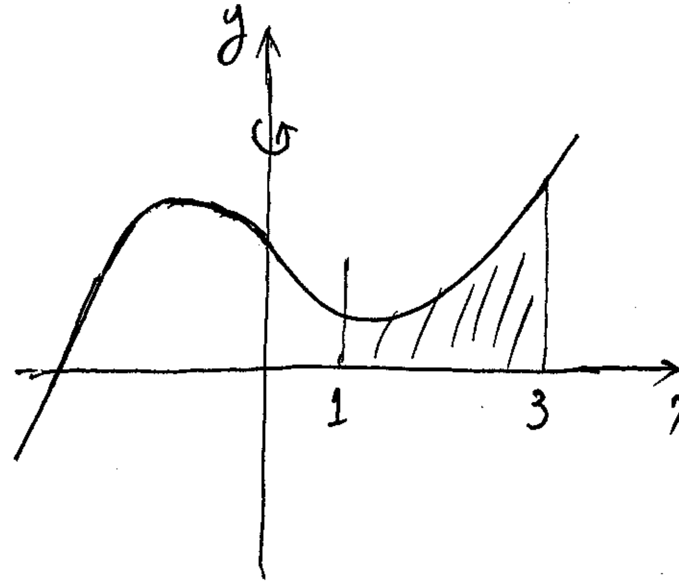
This holds by

x^3-4x+4

$\geq x^2-4x+4$

$= (x-2)^2$

≥ 0.



Second, integrate by the method of cylindrical shells. Get

$$V = \int_1^3 2\pi x y \, dx$$

$$= \int_1^3 2\pi x (x^3-4x+4) \, dx$$

$$= 2\pi \int_1^3 (x^4-4x^2+4x) \, dx$$

$$= 2\pi \left(\frac{1}{5}x^5 - \frac{4}{3}x^3 + 2x^2 \right) \Big|_1^3$$

$$= \left(\frac{484}{5} - \frac{208}{3} + 32 \right) \pi .$$