Solution of the Sample

1(1): The char. eq is

$$r^2-6r+9=0$$

It has a double root $r=3$.

General solution

$$y=c_1 e^{3t}+c_2 t e^{3t}, \qquad c_1, c_2 \in \mathbb{R}.$$

1(2): The char. eq is

$$r^3+4r^2+3r=0$$

$$r(r+1)(r+3)=0$$

It has 3 simple roots $r=0,-1,-3$.

General solution

$$y=c_1+c_2 e^{-t}+c_3 e^{-3t}, \qquad c_1, c_2, c_3 \in \mathbb{R}.$$

1(3): The char. eq is

$$r^3+r^2+r+1=0$$

$$(r^2+1)(r+1)=0$$

It has 3 simple roots

$$r=-1, i, -i$$

General solution $y=c_1 e^{-t}+c_2\cos t+c_3\sin t$.

$c_1, c_2, c_3 \in \mathbb{R}$.

②

2(1): $r=3$ is a double root of the char. eq.

Then

$$y_p = t^2(At+B)\,e^{3t} = (At^3+Bt^2)e^{3t}.$$

2(2): $3+i$ is not a root of the char. eq.

then

$$y_p = t\,e^{3t}(A\cos t + B\sin t) + e^{3t}(C\cdot\cos t + D\cdot\sin t).$$
$$= (At+C)\,e^{3t}\cos t + (Bt+D)\,e^{3t}\sin t.$$

2(3): $\pm i$ are simple roots of the char. eq.

Then

$$y_p = t(At^2+Bt+C)\cos t + t(Dt^2+Et+F)\sin t.$$

3: 1st step: The corresponding homogeneous equation

$$y'' + 4y' + 7y = 0$$

has char. eq.

$$r^2+4r+7=0,$$

which has complex roots

$$r = -2 \pm \sqrt{3}\,i.$$

The solution space has a basis

$$e^{-2t}\cos(\sqrt{3}t), \quad e^{-2t}\sin(\sqrt{3}t).$$

2nd step: A particular solution of the non-homog. eq. has the form

$$y_p = A e^{2t} + B.$$

Plug in. Get

$$y_p'' + 4y_p' + 7y_p = 19Ae^{2t} + 7B$$

Then $\begin{cases} 19A = 1 \\ 7B = 1 \end{cases}$, $\begin{cases} A = \frac{1}{19} \\ B = \frac{1}{7} \end{cases}$.

$$y_p = \frac{1}{19} e^{2t} + \frac{1}{7}$$

3rd step: General sol

$$y = \frac{1}{19}e^{2t} + \frac{1}{7} + c_1 e^{-2t}\cos(\sqrt{3}t) + c_2 e^{-2t}\sin(\sqrt{3}t).$$

$c_1, c_2 \in \mathbb{R}$.

4: Set

$$\begin{cases} x_1 = y \\ x_2 = y' \\ x_3 = y'' \\ x_4 = z \\ x_5 = z' \end{cases},$$

Get

$$\begin{cases} x_3' - x_2 + x_5 + 2x_1 + 5x_4 = 0 \\ x_5' + x_1 - 3x_5 = 0 \\ x_1' = x_2 \\ x_2' = x_3 \\ x_4' = x_5 \end{cases}$$

Rewrite the equations as

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = -2x_1 + x_2 - 5x_4 - x_5 \\ x_4' = x_5 \\ x_5' = -x_1 + 3x_5 \end{cases}$$

Then we have $\vec{x}' = A\vec{x}$ with

$$\vec{x} = \begin{pmatrix} y \\ y' \\ y'' \\ z \\ z' \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -2 & 1 & 0 & -5 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

5: For the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & -14 & 7 \end{pmatrix},$$

we have

$$\det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 8 & -14 & 7-\lambda \end{pmatrix}$$

$$= -\lambda \begin{vmatrix} -\lambda & 1 \\ -14 & 7-\lambda \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ 8 & 7-\lambda \end{vmatrix} \quad \text{(expand by the first row)}$$

$$= -\lambda(\lambda^2 - 7\lambda + 14) + 8$$

$$= -(\lambda^3 - 7\lambda^2 + 14\lambda - 8).$$

Observe: $\lambda = 1$ is a root.

This gives a decomposition:

$$\begin{aligned}
&\lambda^3 - 7\lambda^2 + 14\lambda - 8 \\
&= (\lambda^3 - 7\lambda^2 + 6\lambda) + (8\lambda - 8) \\
&= \lambda(\lambda-1)(\lambda-6) + 8(\lambda-1) \\
&= (\lambda-1)(\lambda^2 - 6\lambda + 8) \\
&= (\lambda-1)(\lambda-2)(\lambda-4).
\end{aligned}$$

For the eigenspaces, have:

$\underline{\lambda = 1}$: $A - I = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 8 & -14 & 6 \end{pmatrix}$

null space spanned by $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$\underline{\lambda = 2}$: $A - 2I = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 8 & -14 & 5 \end{pmatrix}$

null space spanned by $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$.

$\underline{\lambda = 4}$: $A - 4I = \begin{pmatrix} -4 & 1 & 0 \\ 0 & -4 & 1 \\ 8 & -14 & 3 \end{pmatrix}$

null space spanned by $\begin{pmatrix} 1 \\ 4 \\ 16 \end{pmatrix}$.

Finally, the expression for a general sol. is

$$\vec{x} = c_1 e^{t}\begin{pmatrix}1\\1\\1\end{pmatrix} + c_2 e^{2t}\begin{pmatrix}1\\2\\4\end{pmatrix} + c_3 e^{4t}\begin{pmatrix}1\\4\\16\end{pmatrix}.$$

Remark: The computation of the eigenvalues is rather long. In the final exam, the characteristic equation will be given.

6: By row operations,

$$A \xrightarrow[(R4)+2(R1)]{(R2)-5(R1)} \begin{pmatrix} 1 & 3 & 4 & -1 \\ 0 & -14 & -20 & 7 \\ 0 & -2 & 1 & 1 \\ 0 & 8 & 9 & -4 \end{pmatrix} \xrightarrow[(R4)+4\cdot(R3)]{(R2)-7\cdot(R3)} \begin{pmatrix} 1 & 3 & 4 & -1 \\ 0 & 0 & -27 & 0 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & 13 & 0 \end{pmatrix}$$

$$\xrightarrow[\text{clear 3rd column}]{(R4)/13} \begin{pmatrix} 1 & 3 & 0 & -1 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[(R1)-3\cdot(R2)]{(R2)/(-2)} \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Pivot columns are the 1st, 2nd and 3rd columns

Thus a basis of the column space is

$$\begin{pmatrix}1\\5\\0\\-2\end{pmatrix}, \begin{pmatrix}3\\1\\-2\\2\end{pmatrix}, \begin{pmatrix}4\\0\\1\\1\end{pmatrix}.$$

For the null space, the equation

$$\begin{cases} x_1 + \frac{1}{2}x_4 = 0 \\ x_2 - \frac{1}{2}x_4 = 0 \\ x_3 = 0 \end{cases}$$

has solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}x_4 \\ \frac{1}{2}x_4 \\ 0 \\ x_4 \end{pmatrix} = x_4 \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 1 \end{pmatrix}.$$

Then null space has basis $\begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$.

7: Let a_{ij} be the (i,j)-entry of A.

Then $A + A^T = 0$ means $a_{ij} + a_{ji} = 0$, for any i, j.

Thus A takes the form

$$\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}$$

Have 6 free symbols in A.

A basis of V is given by matrices

$$B_{12},\ B_{13},\ B_{14},\ B_{23},\ B_{24},\ B_{34}.$$

Here B_{ij} is the 4×4 matrix with

$$\begin{cases} (i,j)\text{-entry} & 1 \\ (j,i)\text{-entry} & -1 \\ \text{other entry} & 0 \end{cases}$$

Hence, $\dim V = 6$.

8. Char. eq. is

$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 1 & 3-\lambda \end{vmatrix} \qquad \begin{pmatrix} (R1)-(3-\lambda)\cdot(R2) \\ (R3)-(R2) \end{pmatrix}$$

$$= \begin{vmatrix} 0 & 1-(3-\lambda)^2 & \lambda-2 \\ 1 & 3-\lambda & 1 \\ 0 & \lambda-2 & 2-\lambda \end{vmatrix} \qquad \text{(expand by 1st column)}$$

$$= -\begin{vmatrix} (\lambda-2)(4-\lambda) & \lambda-2 \\ \lambda-2 & 2-\lambda \end{vmatrix}$$

$$= -(\lambda-2)^2 \begin{vmatrix} 4-\lambda & 1 \\ 1 & -1 \end{vmatrix}$$

$$= -(\lambda-2)^2(\lambda-5)$$

eigenvalues $\lambda = 2$, $\lambda = 5$.

For eigenspaces:

(1). $\lambda = 2$ gives $A - 2I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

eigenspace gen. by $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

By Gram-Schmidt, it gives orthonormal basis

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \end{pmatrix}.$$

(2). $\lambda = 5$ gives

$$A - 5I = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

eigenspace spanned by $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, or use $\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$.

Finally, get diagonalization

$$A = PDP^{-1}$$

$$D = \begin{pmatrix} 2 & & \\ & 2 & \\ & & 5 \end{pmatrix}, \quad P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}.$$

The exponential

$$e^A = P\, e^D\, P^{-1}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} e^2 & & \\ & e^2 & \\ & & e^5 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}.$$

(Leave this as the final answer.)