THE GROSS–ZAGIER FORMULA (NUMBER THEORY SEMINAR, BERKELEY, FALL 2015)

XINYI YUAN

The goal of this series of talks is to study the Gross–Zagier (GZ) formula and related results. The main results we will cover this semester are:

- the original GZ formula [GZ] (lectures 2-6),
- the result of Kolyvagin [Gr, Ru] and Wei Zhang [Zh],
- Waldspurger's period formula [Wa],
- the GZ formula of Yuan-Zhang-Zhang [YZZ].

Lectures 2-5 are basic and slow, while the remaining ones are more intense. If we continue the topic next semester, we may cover:

- the p-adic Gross–Zagier formula,
- the Gan–Gross–Prasad conjecture,
- introduction to Kudla's program,
- the higher derivative formula of Yun-Zhang over function fields.

Lecture 1. *Introduction and organization*, Xinyi Yuan, Aug 26. I will outline the materials and distribute the weekly lectures to volunteers.

Lecture 2. *Basics on modular curves.* The goal is to review some basic notions on modular curves for congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. It should include canonical models (cf. [Mi, §3]), integral models, moduli interpretations and CM points.

Lecture 3. *The modularity theorem.* The goal is the correspondence between elliptic curves and modular forms. One direction is the Eichler–Shimura correspondence, and the other direction is the modularity theorem of Wiles et al. A basic reference is [DS]. The talk may also include some ideas of the modularity lifting.

Lecture 4. *Rankin–Selberg L-function.* Introduce the L-function $L(f, \chi, s)$ in [GZ] and its integral interpretation. Deduce its analytic continuation and functional equation.

Lecture 5. *Neron–Tate height.* Introduce the Neron–Tate height pairing on curves, Arakelov's intersection theory on arithmetic surfaces, and the arithmetic Hodge index theorem. It should cover [YZZ, §7.1.1-7.1.4].

Lecture 6. *GZ formula and its consequences.* This is the closing talk on the original GZ. Cover most of [GZ, §I,§V] and sketch the main idea of proof.

Lecture 7. *Kolyvagin's work.* Introduce Kolyvagin's work. Follow either [Gr] or [Ru].

Lecture 8. *Wei Zhang's work*. Introduce the result of [Zh] on Kolyvagin's conjecture.

Lecture 9. *The formula of Waldspurger*. Derivative formulas are always accompanied by analogous period formulas. Introduce the period formula of Waldspurger. Cover [YZZ, §1.4]. This is in terms of automorphic representations.

Lecture 10. *The formula of YZZ*. Introduce the formula. Cover [YZZ, §1.2-1.3].

References

[Gr] B. Gross: Kolyvagin's work on modular elliptic curves (available at `http://wstein.org/papers/bib/gross-kolyvagins_work_on_modular_elliptic_curves.pdf`).

[Ru] K. Rubin: The work of Kolyvagin on the arithmetic of elliptic curves.

[GZ] B. Gross; D. Zagier: *Heegner points and derivatives of L-series.* Invent. Math. 84 (1986), no. 2, 225–320.

[DS] F. Diamond; J. Shurman: A first course in modular forms. Vol. 228. Springer Science & Business Media, 2006.

[DZ] H. Darmon; S. Zhang: *Heegner points and Rankin L-series.* Math. Sci. Res. Inst. Publ., 49. Cambridge Univ. Press, Cambridge, 2004. (available at `http://library.msri.org/books/Book49/contents.html`).

[Mi] J. Milne: Canonical models of Shimura curves. (available at `http://www.jmilne.org/math/articles/2003a.pdf`).

[Wa] J. Waldspurger: *Sur les valeurs de certaines fonctions L automorphes en leur centre de symétrie.* Compositio Math. 54 (1985), no. 2, 173–242.

[YZZ] X. Yuan, S. Zhang, W. Zhang: *The Gross–Zagier Formula on Shimura Curves.* Annals of Mathematics Studies, No. 184, Princeton University Press, 2013.

[Zh] W. Zhang: Selmer groups and the indivisibility of Heegner points. Cambridge Journal of Math., Vol. 2 (2014), No. 2, 191–253.