

**ARITHMETIC OF ELLIPTIC CURVES  
OVER FUNCTION FIELDS  
(NUMBER THEORY SEMINAR, BERKELEY, SPRING 2015)**

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The BSD conjecture, usually stated for elliptic curves over number fields, can be similarly formulated for abelian varieties over global fields. In fact, when restricted to global function fields, much more is known about the conjecture. The most striking known result is that the BSD conjecture for (all) abelian varieties over global function fields is equivalent to the Tate conjecture for (all) surfaces over finite fields. For quick overview of the theory, we refer to Douglas Ulmer's surveys [U1, U2].

The goal of this seminar is to study the BSD conjecture over global function fields, and the especially its connection to the Tate conjecture. The following is the proposed lecture series, to be mostly presented by graduate students. The references I give below are not standard, but very accessible to beginners.

Lecture 1. *Introduction and organization*, Xinyi Yuan, Jan 28. I will sketch what are known over function fields, and outline the materials we will focus on. Then we will distribute the incoming weekly lectures to volunteers.

Lecture 2. *Statement of the full BSD conjecture*. The goal is to state the full BSD conjecture for elliptic curves over global fields. Define all the terms in the conjecture. In the case of elliptic curves over function fields, briefly recall the analytic property of the L-functions and the finiteness of the Mordell–Weil groups. Introduce what is known. Follow [U1, Lecture 1], except that the full BSD conjecture should be found in [U2, Conjecture 6.2.5].

Lecture 3. *Étale cohomology I: introduction*. It may takes one hour to even define étale cohomology (assuming everything in Hartshorne's book), but the purpose here is to convince us the existence of such a theory. Introduce three examples without proofs: (1) Identification with the usual (Zariski) sheaf cohomology when the étale sheaf is quasi-coherent. (2) Identification with Galois cohomology when the base is the spectrum of a field. (3) Interpretation of  $H^1(S, G)$  as the group of  $G$ -torsors. Recommend [Po, Chap. 6] to beginners. See also [Mi1, Chap. 3].

Lecture 4. *Étale cohomology II: Weil conjecture*. Tell this fabulous story in one hour. State all the Weil's conjecture. Introduce  $\ell$ -adic cohomology. Deduce the rationality of Weil's zeta function from it. Again, recommend [Po, Chap. 7] to beginners. See also [Mi1, VI.12].

Lecture 5. *Introduction to the Tate conjecture*. Present in the generality of any (smooth and projective) variety over finite fields. See [Mi2].

Lecture 6. *BSD vs Tate: rank part.* Explore the relation between both sides of the conjectures. For this lecture, only do the rank part. Give a full proof of [U11, Lecture 3, Thm. 8.1(1)].

Lecture 7. *BSD vs Tate: obstruction part.* Introduce the isomorphism between the Tate–Shafarevich group and the Brauer group. Two tasks in this hour: (1) Introduce spectral sequences. See [Po, §6.7] for example. (2). Sketch an idea of the proof, which should be less involved than [U12, §5.3.1], but should be complete assuming the fibration is everywhere smooth.

Lecture 8. *Brauer group of curves: class field theory.* Talk for the role of Brauer groups in both the local class field theory and the global class field theory. This is in fact a more basic topic, but it illustrates the significance of Brauer groups even in this one-dimensional case. See [CF, VI-VII].

Lecture 9. *Tate conjecture: homomorphisms between abelian varieties.* Introduce Tate’s proof of his conjecture in this case. See [Ta].

Lecture 10. *Tate conjecture: K3 surfaces.* This case has been settled recently by the efforts of many people. Give an introduction. I will figured out which paper to focus on.

#### REFERENCES

- [CF] J.W.S. Cassels, A. Fröhlich, Algebraic Number Theory, (Eds), Algebraic Number Theory, Academic Press, 1967.
- [Mi1] J. Milne, Lectures on Etale Cohomology, available at <http://www.jmilne.org/math/CourseNotes/LEC.pdf>.
- [Mi2] J. Milne, The Tate conjecture over finite fields (AIM talk), available at <http://www.jmilne.org/math/articles/2007e.pdf>.
- [Po] B. Poonen, Rational points on varieties, available at <http://www-math.mit.edu/~poonen/papers/Qpoints.pdf>.
- [Ta] J. T. Tate. Endomorphisms of abelian varieties over finite fields. Invent. Math., 2: 134–144, 1966.
- [U11] D. Ulmer, Elliptic curves over function fields, available at <http://people.math.gatech.edu/~ulmer/research/papers/2011.pdf>.
- [U12] D. Ulmer, Curves and Jacobians over function fields, available at <http://people.math.gatech.edu/~ulmer/research/papers/2014b.pdf>.