

MATH 54 MIDTERM II, XINYI YUAN
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(7 PAGES)

Problem Number	1	2	3	4	5	6	Total
Score							

YOUR NAME: _____

Last Name of Your Section Instructor (please circle): Agrawal, Dalal,
Drouot, Gleason, Miller, Yott, Yuan, Zhang

Time of your discussion section: _____

1. (5 points) Find the rank of each of the following matrices. Explain your results briefly.

(a) The 4×4 matrix

$$\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 5 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

Sol: An echelon form is

$$\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The rank is 3.

(b) The 5×3 matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \\ 13 & 14 & 15 \end{pmatrix}.$$

Sol. By the row operations $(R_2)-(R_1)$, $(R_3)-(R_1)$, $(R_4)-(R_1)$ and $(R_5)-(R_1)$, get

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 6 & 6 & 6 \\ 9 & 9 & 9 \\ 12 & 12 & 12 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad \text{Then Rank} = 2.$$

(c) For $a < b < c$, the 3×3 matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}.$$

Sol. Take $(R_2)-a \cdot (R_1)$, $(R_3)-a^2 \cdot (R_1)$. Get

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & (a+b)(b-a) & (a+c)(c-a) \end{pmatrix} \quad (\text{Note the change})$$

Take $(R_3)-(a+b) \cdot (R_2)$, get

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & (c-b)(c-a) \end{pmatrix}.$$

The diagonal entries are nonzero by $a < b < c$.

So the rank is 3.

2. (5 points) Let V be the vector space of all polynomials $f(t)$ of degrees less than 5 and satisfying $f(-t) = f(t)$. Let $T: V \rightarrow V$ be the linear transformation defined by $T(f(t)) = f(t) - f(0)$.

(a) Find the dimension of V .

(b) Describe the kernel and the range of T .

(Note: We have assumed that V is a vector space and that T is a linear transformation. You are not required to prove these two statements, and you can use them freely.)

Sol. (a). Write $f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4$.
Then $f(-t) = a_0 - a_1 t + a_2 t^2 - a_3 t^3 + a_4 t^4$.

The relation $f(t) = f(-t)$ gives

$a_0 = a_0, a_1 = -a_1, a_2 = a_2, a_3 = -a_3, a_4 = a_4$.
So $a_1 = a_3 = 0, a_0, a_2, a_4$ arbitrary.

A basis of V is $\{1, t^2, t^4\}$.

$\dim V = 3$.

(b). For $f(t) = a_0 + a_2 t^2 + a_4 t^4$,

$$T(f(t)) = f(t) - f(0) = a_2 t^2 + a_4 t^4$$

If $T(f(t)) = 0$, then $a_2 t^2 + a_4 t^4 = 0, a_2 = a_4 = 0$

So $\ker(T) = \text{Span}\{1\} = \{a_0\}$

$$f(t) = a_0$$

Since a_2, a_4 can be arbitrary,

$$\text{range}(T) = \text{Span}\{t^2, t^4\} = \{a_2 t^2 + a_4 t^4\}$$

3. (5 points) Find all values of a such that the matrix

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & a \end{pmatrix}$$

is diagonalizable.

Sol: A is upper triangular

Then the eigenvalues are $1, 2, a$.

①. If $a \neq 1, 2$, all eigenvalues are distinct
 A is diagonalizable

②. If $a = 1$, then $\lambda = 1$ has multiplicity 2.

The eigenspace is the solution set of

$$(A - I)\vec{x} = 0 \Rightarrow \begin{pmatrix} 0 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{x} = 0$$

This gives $\text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$, which is 1-dimensional

We have $1 < 2$. Not diagonalizable

③. If $a = 2$, then $\lambda = 2$ has multiplicity 2

The eigenspace is the solution set of

$$(A - 2I)\vec{x} = 0 \Rightarrow \begin{pmatrix} -1 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{x} = 0$$

This gives $\text{span}\left\{\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$, which is 2-dim'l.

We have $2 = 2$. The root $\lambda = 1$ has multiplicity 1, which is not a problem.

So A is diagonalizable in this case.

Conclusion: A is diagonalizable $\Leftrightarrow \lambda \neq 1$.

4. (5 points) Let W be the subspace of \mathbb{R}^4 given by $x_1 + x_2 + x_3 + x_4 = 0$. Compute the orthogonal projection $\text{proj}_W(\vec{v})$ of the vector $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ to the space W .

Sol: By definition,

$$W^\perp = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Then

$$\begin{aligned} & \text{proj}_W(\vec{v}) \\ &= \vec{v} - \text{proj}_{W^\perp}(\vec{v}) \\ &= \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} - \frac{1+2+3+4}{1+1+1+1} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} -3 \\ -1 \\ 1 \\ 3 \end{pmatrix}. \end{aligned}$$

Remark: Another way is to find a basis of W by the usual way of solving equations, change it to an orthogonal basis by Gram-Schmidt, and then compute $\text{proj}_W(\vec{v})$ directly using this orthogonal basis.

5. (5 points) Let A be a 2×2 matrix which has two distinct real eigenvalues. Assume that $A^3 + A^2 - A - I_2 = O_2$. Find all possible forms of the matrix A^2 . (Note: I_2 denotes the 2×2 identity matrix, and O_2 denotes the 2×2 matrix whose entries are all 0.)

Sol: A has 2 distinct real eigenvalues,

$\Rightarrow A$ is diagonalizable

write $A = PDP^{-1}$, $D = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$.

Then $A^3 + A^2 - A - I_2$

$$= PD^3P^{-1} + PD^2P^{-1} - PDP^{-1} - I$$

$$= P(D^3 + D^2 - D - I)P^{-1}$$

Since P is invertible, have

$$D^3 + D^2 - D - I = 0,$$

which is $\begin{pmatrix} f(\lambda_1) \\ f(\lambda_2) \end{pmatrix} = 0$

where $f(t) = t^3 + t^2 - t - 1 = (t+1)^2(t-1)$.

Then $f(\lambda_1) = f(\lambda_2) = 0$, which gives $\lambda_1, \lambda_2 = \pm 1$.

Hence, $D^2 = \begin{pmatrix} \lambda_1^2 & \\ & \lambda_2^2 \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = I_2$,

$$A^2 = PD^2P^{-1} = PIP^{-1} = I.$$

6. (5 points) Let A and B be $n \times n$ matrices, and assume that B is invertible. Is it always true that $\text{rank}(AB) = \text{rank}(A)$? (Note: If your answer is *yes*, give a reason. If your answer is *no*, give a counter-example.)

Sol: The answer is yes.

Consider A, B, AB as linear transformations from \mathbb{R}^n to \mathbb{R}^n by multiplications to vectors. Since B is invertible, $B\vec{v}$ can be any vector in \mathbb{R}^n . Then

$$\begin{aligned} \text{range}(AB) &= \{ A(B\vec{v}) \mid \vec{v} \in \mathbb{R}^n \} \\ &= \{ A\vec{w} \mid \vec{w} \in \mathbb{R}^n \} \\ &= \text{range}(A). \end{aligned}$$

Count the dimensions. Get $\text{rank}(AB) = \text{rank}(A)$.

Rank. There are many ways to do it.

For example, by matrix multiplication, have

$$\text{col}(AB) \subset \text{col}(A),$$

b/c columns of AB are lin. comb. of columns of A . Take dimensions, we have

$$\text{rank}(AB) \leq \text{rank}(A).$$

Similarly, have

$$\text{rank}(AB \cdot B^{-1}) \leq \text{rank}(AB). \quad (\text{think: } C=AB)$$