Reductions of abelian varieties and K3 surfaces

Ananth N. Shankar and Yunqing Tang

April 2, 2024

Abstract

This article is a survey of our work (joint with Davesh Maulik, Arul Shankar, and Salim Tayou) on arithmetic intersection theory on GSpin Shimura varieties with applications to abelian varieties, K3 surfaces, and the ordinary Hecke orbit conjecture.

1 Introduction

In [Cha18], François Charles proves the following beautiful theorem by arithmetic intersection theory on the j-line. Let E, E' denote two elliptic curves defined over a number field K. Then there are infinitely many primes v of K such that $E_{\overline{\mathbb{F}}_v}$ and $E'_{\overline{\mathbb{F}}_v}$ are geometrically isogenous, where $\overline{\mathbb{F}}_v$ denotes an algebraic closure of the residue field $\mathbb{F}_v$. Earlier work of Chai and Oort ([CO06]) proves an analogous result in positive characteristic. In a series of papers [ST20, MST22, SSTT22, MST22b] (joint with Davesh Maulik, Arul Shankar, and Salim Tayou), we generalize this work to the setting of GSpin Shimura varieties, and deduce similar applications to the Picard ranks of K3 surfaces and splitting of abelian varieties, both defined over global fields. Our results are in the setting of abelian varieties and K3 surfaces having potentially good reduction everywhere, and the case of bad reduction is finished in [Tay24]. In this paper, we give a survey of our joint work. For brevity, we will state our main results in the setting of number fields in the introduction. The case of positive characteristic and their consequences are discussed Section 4.

Abelian surfaces: Let A/K denote an abelian surface defined over a number field. Then there are infinitely many primes v of K such that $A_{\overline{\mathbb{F}}_v}$ is isogenous to a product of elliptic curves.

K3 surfaces: Let X/K denote a K3 surface over a number field. Then there are infinitely many primes v of K such that the Picard rank of $X_{\overline{\mathbb{F}}_v}$ is greater than the Picard rank of $X_{\overline{K}}$.

We prove all our theorems (both in the number field case, as well as in the case of positive characteristic) using the framework of arithmetic intersection theory. Consider the setting of abelian surfaces. Loosely speaking, A/K gives rise to an $\mathcal{O}_K[1/N]$-valued point of the moduli space of polarized abelian surfaces. The exceptional isomorphism between PSp_4 and SO_5 implies that we may view the moduli space of polarized abelian surfaces as a GSpin Shimura variety! The sub-locus of non-simple abelian surfaces is a countably infinite union of *special divisors*, each of which is a GSpin Shimura variety in its own right. In order to show that $A_{\overline{\mathbb{F}}_v}$ is not simple for infinitely many primes v, it suffices to show that the moduli point intersects this infinite union of special divisors at infinitely many closed points. The setting of K3 surfaces is the same — the ambient space, the moduli space of quasi-polarized K3 surfaces whose Picard groups contain a given quadratic lattice, is a higher dimensional GSpin Shimura variety, and the (countably infinite family of) *special* *divisors* parameterizing K3 surfaces with extra line bundles are also GSpin Shimura varieties. As will be clear, the higher dimensional cases require significantly new ideas as compared to the lower dimensional cases. The overarching method is to use intersection-theoretic methods to prove that an arithmetic curve (a number field valued point spreads out to an an arithmetic curve) mapping to a GSpin Shimura variety has the property that infinitely many closed points map to some special divisors.

The characteristic p settings of our theorems have an unexpected consequence to the ordinary Hecke orbit problem (conjectured by Chai and Oort). Roughly speaking, we prove that the prime-to-p Hecke orbit of an ordinary point in a mod p GSpin Shimura variety must be Zariski dense. We remark that Daniel Bragg and Ziquan Yang use this theorem to prove an analogue of the Néron–Ogg–Shafarevich conjecture for K3 surfaces [BY23].

Organization

In Section 2, we outline different heuristics, including one that suggests the existence of unlikely intersections. In Section 3, we review the definitions and terminology for GSpin Shimura varieties and special divisors. In Section 4, we precisely state all our theorems, describe the work of Charles and Chai–Oort, and give detailed outlines of the proofs of our main theorems while focusing on various obstructions and the ideas we use to overcome these difficulties. Finally, in Section 5, we describe conjectures and other existing work related our theorems.

Acknowledgement

First and foremost, it is a great pleasure to thank our co-authors Davesh Maulik, Arul Shankar, and Salim Tayou. We also acknowledge Ching-Li Chai, François Charles, and Frans Oort, whose work inspired every result of [ST20], [SSTT22], [MST22] and [MST22b]. We are also very grateful to Edgar Costa, Elena Mantovan, Peter Sarnak, Jacob Tsimerman, Tonghai Yang, and Shouwu Zhang, for helpful discussions. A.N.S. has been partially supported by the NSF grant DMS-2100436. Y.T. has been partially supported by NSF grant DMS-2231958 and a Sloan Research Fellowship.

2 Heuristics

2.1 Splitting of abelian surfaces

Let $A/\mathbb{Q}$ be an abelian surface. For simplicity, we assume that $\mathrm{End}(A_{\overline{\mathbb{Q}}}) = \mathbb{Z}$. For every prime p, let I_p denote the set of geometric isogeny classes of abelian surfaces defined over $\mathbb{F}_p$. Therefore, for every (large enough) prime p, we obtain an element of I_p by considering the mod p reduction of A. The heuristic that suggests infinitude of primes of split reduction (and indeed, the order of growth of the counting function of such primes) is to treat this assignment as a random map (where each isogeny class $x \in I_p$ is weighted by its size). There are precise expectations as to the size of each isogeny class (see [ST18], [Bha], [Fu23]). Roughly speaking, the size of each ordinary isogeny class is expected to be approximately $p^{\frac{3}{2}\pm\epsilon}$, the size of I_p is also expected to be $p^{\frac{3}{2}\pm\epsilon}$, and the number of non-ordinary isogeny classes is expected to be $O(p^{1+\epsilon})$.[1] The set of $\mathbb{F}_p$-split isogeny classes in I_p is expected to be around $p^{1\pm\epsilon}$. Therefore, one should expect A mod p is split with a probability

[1]Here by the expected size to be X, we mean that the size is bounded below and above by nonzero constant multiples of X. This applies to the discussions in Sections 2.1 and 2.2.

of about $\frac{1}{\sqrt{p}}$. This suggests that there should be infinitely many primes of split reduction, and indeed, the number of split primes less than X should equal $X^{\frac{1}{2}\pm\epsilon}$ for any $\epsilon \geq 0$. Similarly, work of Achter–Howe [AH17] on the number of split abelian surfaces over finite fields also suggests such results.

2.2 Unlikely intersections

For $A/\mathbb{Q}$ as in Section 2.1, consider the question of primes p modulo which A is geometrically isogenous to E^2 for an elliptic curve E over $\overline{\mathbb{F}}_p$. As the number of isogeny classes of elliptic curves over $\mathbb{F}_p$ is approximately $p^{\frac{1}{2}\pm\epsilon}$, one might be tempted to deduce that the probability that $A \bmod p$ is isogenous to E^2 is around $\frac{1}{p}$, which in turn would suggest that the prime counting function should have order of growth $O(\log\log X)$. However, data gathered by Edgar Costa suggests that the number of primes bounded by X modulo which A is geometrically isogenous to E^2 grows as $X^{1/2\pm\epsilon}$! In other words, Costa's data suggests that there are unlikely intersections. The Zilber–Pink conjectures suggest that this behaviour should not be seen over $\mathbb{C}$ — specifically, given a generic family of abelian surfaces A/C, where C is a complex algebraic curve, there ought to be only finitely many points $x \in C$ such that A_x is isogenous to a self-product of complex elliptic curves. We suspect that this discrepancy arises because of abelian surfaces over $\mathbb{F}_p$ of the form $E \times E^\sigma$, where $E/\mathbb{F}_{p^2}$ is an elliptic curve and E^σ is its Frobenius twist – in other words, abelian surfaces which are not isogenous to E^2 over $\mathbb{F}_p$, but which are isogenous to E^2 after passing to a quadratic extension. Indeed, similar counts yield that the number of such isogeny classes should be around $p^{1\pm\epsilon}$.

2.3 Expected Picard rank jumps of K3 surfaces

Let X be a K3 surface over a number field K. The K3 surface X admits a smooth projective model away from a finite set of primes S, which we denote by $\mathcal{X}/\mathcal{O}_{K,S}$. Let $\mathfrak{p} \subset \mathcal{O}_{K,S}$ be a non-zero prime ideal, and let $\overline{\mathbb{F}}_\mathfrak{p}$ be its residue field. There is an injective specialization map between Picard groups $\operatorname{Pic}(X_{\overline{K}}) \to \operatorname{Pic}(\mathcal{X}_{\overline{\mathbb{F}}_\mathfrak{p}})$. Therefore $\operatorname{rk}_\mathbb{Z} \operatorname{Pic}(\mathcal{X}_{\overline{\mathbb{F}}_\mathfrak{p}}) \geq \operatorname{rk}_\mathbb{Z} \operatorname{Pic}(X_{\overline{K}})$.

The work of Charles [Cha14, Thm. 1][2] provided a full description of the geometric Picard rank $\operatorname{rk}_\mathbb{Z} \operatorname{Pic}(\mathcal{X}_{\overline{\mathbb{F}}_\mathfrak{p}})$ at density 1 set of primes $\mathfrak{p}$ (one may need to replace K by a finite extension) in terms of the endomorphism field of the Hodge structure associated to the transcendental part of $H^2_B(X(\mathbb{C}), \mathbb{Q})$.[3] More precisely, let $T \subset H^2_B(X(\mathbb{C}), \mathbb{Q})$ denote the sub-Hodge structure defined to be the orthogonal complement of the Néron–Severi group $NS(X_{\overline{K}}) \subset H^2_B(X(\mathbb{C}), \mathbb{Q})$ with respect to the intersection form. Note that for K3 surfaces, $NS(X_{\overline{K}}) \cong \operatorname{Pic}(X_{\overline{K}})$. The subspace T is usually called the transcendental part of $H^2_B(X(\mathbb{C}), \mathbb{Q})$. By [Zar83, Thms 1.5.1, 1.6], the endomorphism algebra E of the Hodge structure T is either a totally real field or a CM field. We may view T as an E-vector space. Charles's theorem [Cha14, Thm. 1] shows that after possibly replacing K by a finite extension:

1. If E is totally real and $\dim_E T$ is odd, then $\operatorname{rk}_\mathbb{Z} \operatorname{Pic}(\mathcal{X}_{\overline{\mathbb{F}}_\mathfrak{p}}) \geq [E : \mathbb{Q}] + \operatorname{rk}_\mathbb{Z} \operatorname{Pic}(X_{\overline{K}})$ for all primes $\mathfrak{p} \notin S$ and the equality holds for a density 1 set of primes.

[2] One input of the proof is the Tate conjecture for K3 surfaces over finite field, which was proved by Madapusi [MP15], Ito–Ito–Koshikawa [IIK21] and Kim–Madapusi [KMP16, MP20]. Many others have contributed to the conjecture including Charles [Cha16], Maulik [Mau14], and Nygaard–Ogus.[NO85].

[3] One chooses an embedding $K \to \mathbb{C}$ to consider the complex manifold $X(\mathbb{C})$ and the statement/result is independent of the choice of the embedding.

2. If E is either CM or $\dim_E T$ is even, then $\mathrm{rk}_{\mathbb{Z}} \mathrm{Pic}(\mathcal{X}_{\overline{\mathbb{F}}_{\mathfrak{p}}}) = \mathrm{rk}_{\mathbb{Z}} \mathrm{Pic}(X_{\overline{K}})$ for a density 1 set of primes.

In [SSTT22, Thm. 1.1] and [Tay24, Thm. 1.1], it is proved that even in case (2) above, there are infinitely many primes $\mathfrak{p}$ such that $\mathrm{rk}_{\mathbb{Z}} \mathrm{Pic}(\mathcal{X}_{\overline{\mathbb{F}}_{\mathfrak{p}}}) > \mathrm{rk}_{\mathbb{Z}} \mathrm{Pic}(X_{\overline{K}})$. Based on the discussions in [Cha14, §3] and [Zar83, §§1-2], we have

Lemma 2.1. *Let $\mathfrak{p}$ be a prime such that $\mathrm{rk}_{\mathbb{Z}} \mathrm{Pic}(\mathcal{X}_{\overline{\mathbb{F}}_{\mathfrak{p}}}) > \mathrm{rk}_{\mathbb{Z}} \mathrm{Pic}(X_{\overline{K}})$ in case (2) above.*

1. *If E is CM, then $\mathrm{rk}_{\mathbb{Z}} \mathrm{Pic}(\mathcal{X}_{\overline{\mathbb{F}}_{\mathfrak{p}}}) \geq [E : \mathbb{Q}] + \mathrm{rk}_{\mathbb{Z}} \mathrm{Pic}(X_{\overline{K}})$;*
2. *If E is totally real and $\dim_E T$ is even, then $\mathrm{rk}_{\mathbb{Z}} \mathrm{Pic}(\mathcal{X}_{\overline{\mathbb{F}}_{\mathfrak{p}}}) \geq 2[E : \mathbb{Q}] + \mathrm{rk}_{\mathbb{Z}} \mathrm{Pic}(X_{\overline{K}})$.*

Proof. [Zar83, Thms. 2.2.1, 2.3.1] described the Hodge group (i.e., the special Mumford–Tate group) of X acting on T as follows. Let $[-,-]$ denote (-1) times the intersection form on T, which is a nondegenerate symmetric bilinear form with signature $(\dim_{\mathbb{Q}} T, 2)$. If E is CM, we pick a totally imaginary element $\zeta \in E$ and then there exists a unique (nondegenerate) Hermitian form $(-,-)$ on T viewed as an E-vector space such that $[-,-] = \mathrm{Tr}_{E/\mathbb{Q}}(\zeta(-,-))$.[4] Let E^+ denote the maximal totally real subfield of E. The Hodge group acting on T is $\mathrm{Res}_{E^+/\mathbb{Q}} \mathrm{U}(T, (-,-))$. If E is totally real, then there exists a unique (nondegenerate) symmetric bilinear form $(-,-)$ on T viewed as an E-vector space such that $[-,-] = \mathrm{Tr}_{E/\mathbb{Q}}(-,-)$. The Hodge group acting on T is $\mathrm{Res}_{E/\mathbb{Q}} \mathrm{SO}(T, (-,-))$.

By Tankeev [Tan90, Tan95], the Mumford–Tate conjecture holds for K3 surfaces and thus after passing to a finite extension of K, the algebraic ℓ-adic monodromy group (i.e., the Zariski closure of the image of the Galois representation) on $T_{\mathbb{Q}_\ell} \subset H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)(1)$ is either $\mathrm{Res}_{E^+/\mathbb{Q}} \mathrm{U}(T_{\mathbb{Q}_\ell}, (-,-))$ or $\mathrm{Res}_{E/\mathbb{Q}} \mathrm{SO}(T_{\mathbb{Q}_\ell}, (-,-))$. Here $T_{\mathbb{Q}_\ell}$ denotes the orthogonal complement of $\mathrm{Pic}(X_{\overline{K}}) \otimes \mathbb{Q}_\ell \subset H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)(1)$ with respect to the cup product; via the Betti-étale comparison isomorphism, $T_{\mathbb{Q}_\ell}$ is identified with $T \otimes \mathbb{Q}_\ell$.

By the Tate conjecture for K3 surfaces, $\mathrm{rk}_{\mathbb{Z}} \mathrm{Pic}(\mathcal{X}_{\overline{\mathbb{F}}_{\mathfrak{p}}}) - \mathrm{rk}_{\mathbb{Z}} \mathrm{Pic}(X_{\overline{K}})$ is exactly the multiplicity of the eigenvalues of $\mathrm{Frob}_{\mathfrak{p}}$ acting on $T_{\mathbb{Q}_\ell}$ with values being roots of unity. By the shape of the algebraic ℓ-adic monodromy above, it must be at least $[E : \mathbb{Q}]$ if E is CM and $2[E : \mathbb{Q}]$ if E is totally real and $\dim_E T$ is even. □

We expect the equalities in Lemma 2.1 holds at the majority of the infinitely many places $\mathfrak{p}$ satisfying $\mathrm{rk}_{\mathbb{Z}} \mathrm{Pic}(\mathcal{X}_{\overline{\mathbb{F}}_{\mathfrak{p}}}) > \mathrm{rk}_{\mathbb{Z}} \mathrm{Pic}(X_{\overline{K}})$. First consider the CM case. The heuristics are that we may view X as a K-point in a unitary Shimura variety associated to a Hermitian space over E/E^+. The above change of geometric Picard ranks happens exactly when the reduction of X lies on (the natural integral model of) a special divisor but not a codimension 2 special cycle. We expect the majority of reductions lying on special divisors to not lie on codimension 2 special cycles.[5] In the totally real case, the ambient Shimura variety containing the moduli point associated to X is a GSpin Shimura variety associated to an even dimensional quadratic vector space over E. An $\overline{\mathbb{F}}_p$-valued point of such a Shimura variety ought not to be contained in any special divisor. On the other hand, every $\overline{\mathbb{F}}_p$-valued point of a GSpin Shimura variety associated to an odd dimensional

[4] It is more canonical to work with the skew-Hermitian form $\zeta(-,-)$, which is obtained from the above definition without choosing a ζ as in [Zar83]; the choice is harmless as we only consider the associated unitary group here. We choose to use the Hermitian form rather than the skew-Hermitian form to follow the convention of many related work in literature on unitary Shimura varieties.

[5] Here we are vague about subtleties between the actual dimension and virtual dimension. By a codimension 2 special cycle we mean the special cycle obtained from a rank 2 sublattice in the Hermitian space.

quadratic lattice is contained in a special divisor.[6] Therefore if the reduction of the moduli point associated to X lies on a special divisor mod $\mathfrak{p}$, it must lie on a co-dimension 2 special cycle. But a codimension 2 special cycle is a GSpin Shimura variety associated to an even dimensional quadratic space in its own right, and given that a random $\overline{\mathbb{F}}_p$-valued point of such a Shimura variety ought not to be contained in a special divisor, we expect that the majority of reductions do not lie on special cycles associated to rank 3 or higher lattices.

We end this subsection by providing two examples to show that the bounds in Lemma 2.1 are sharp. For (1), take X to be the Kummer surface associated to A^2, where A is an elliptic curve with complex multiplication by an imaginary quadratic field F. In this case, $\mathrm{rk}_{\mathbb{Z}} \mathrm{Pic}(X_{\overline{K}}) = 20$, T is a 2-dimensional $\mathbb{Q}$-vector space and its endomorphism algebra (as a Hodge structure) is F. At the primes $\mathfrak{p}$ such that A has ordinary reduction, $\mathrm{rk}_{\mathbb{Z}} \mathrm{Pic}(\mathcal{X}_{\overline{\mathbb{F}}_{\mathfrak{p}}}) = 20$. At the primes $\mathfrak{p}$ where A has supersingular reduction, $\mathrm{rk}_{\mathbb{Z}} \mathrm{Pic}(\mathcal{X}_{\overline{\mathbb{F}}_{\mathfrak{p}}}) = 22 = 20 + [F : \mathbb{Q}]$. Note that this is also an example indicating that a finite extension is necessary in Charles's theorem. If $F \not\subset K$, then $\mathrm{rk}_{\mathbb{Z}} \mathrm{Pic}(\mathcal{X}_{\overline{\mathbb{F}}_{\mathfrak{p}}}) = \mathrm{rk}_{\mathbb{Z}} \mathrm{Pic}(X_{\overline{K}})$ only for a set of primes having density 1/2. However, the set of primes of the finite extension FK/F modulo which A has ordinary reduction has density 1.

For (2), take X to be the Kummer surface associated to an abelian surface A such that $\mathrm{End}^0(A_{\overline{K}})$ is a real quadratic field (or product of two non-geometrically-isogenous elliptic curves). In this case, $\mathrm{rk}_{\mathbb{Z}} \mathrm{Pic}(X_{\overline{K}}) = 18$, T is a 4-dimensional $\mathbb{Q}$-vector space, and its endomorphism algebra (as a Hodge structure) is $\mathbb{Q}$. If $\mathfrak{p}$ is a prime such that $A \bmod \mathfrak{p}$ is an ordinary abelian surface and $\mathrm{rk}_{\mathbb{Z}} \mathrm{Pic}(\mathcal{X}_{\overline{\mathbb{F}}_{\mathfrak{p}}}) > \mathrm{rk}_{\mathbb{Z}} \mathrm{Pic}(X_{\overline{K}})$, then $\mathrm{rk}_{\mathbb{Z}} \mathrm{Pic}(\mathcal{X}_{\overline{\mathbb{F}}_{\mathfrak{p}}}) = 20 = 18 + 2[\mathbb{Q} : \mathbb{Q}]$. By a theorem of Sawin [Saw16], A has ordinary reduction at a density 1 set of primes over K. We also expect that most primes in the set $\{\mathfrak{p} : \mathrm{rk}_{\mathbb{Z}} \mathrm{Pic}(\mathcal{X}_{\overline{\mathbb{F}}_{\mathfrak{p}}}) > \mathrm{rk}_{\mathbb{Z}} \mathrm{Pic}(X_{\overline{K}})\}$ are primes of ordinary reduction for A. Therefore, the bound in Lemma 2.1 ought to be sharp.

3 Preliminaries

In this section, we review basic definitions, terminology, and notation for GSpin Shimura varieties, special endomorphisms, and special divisors and their extensions to toroidal compactifications that we need in the rest of the paper.

Let (L, Q) be a quadratic $\mathbb{Z}$-lattice of signature $(b, 2)$, $b \geq 1$. Let $V := L \otimes_{\mathbb{Z}} \mathbb{Q}$ and let $[-,-]$ denote the bilinear form on V given by $[x, y] = Q(x + y) - Q(x) - Q(y)$. Let Ω (resp. Ω_0) denote the finite set of primes p such that $L \otimes \mathbb{Z}_p$ is not a maximal (resp. self-dual) lattice in $V \otimes \mathbb{Q}_p$ over which Q is $\mathbb{Z}_p$-valued. By definition, $\Omega \subset \Omega_0$. Let $\mathbb{Z}_\Omega$ denote the ring of Ω-integers. We recall the canonical integral model of the GSpin Shimura variety associated to (L, Q) over $\mathrm{Spec}\, \mathbb{Z}_\Omega$ and the definition of special divisors. The main references are [MP16], [AGHMP18, §4], and [HP20, §§4,6] — see also [SSTT22, §2] and [MST22b, §2] for a brief summary. We also recall the integral models of toroidal compactifications of GSpin Shimura varieties. The main references are [HP20, §§2, 4, 8] and [BZ21, §§2-3] — see also [Tay24, §2] for a brief summary and a discussion about the extension of special divisors to the boundary.

[6] The group theory that underlies this phenomenon is as follows. Consider a maximal torus defined over E inside an odd-dimensional orthogonal group over E. Then the set of long roots associated to this maximal torus form a Galois-stable set, and the group generated by this maximal torus and root-groups associated to the long roots is an E-orthogonal group of type D. Indeed, this may be viewed as the geometric and group theoretic explanation of Charles's theorem (1).

3.1. Let $G := \mathrm{GSpin}(V, Q)$ be the group of spinor similitudes of V, which is a reductive group over $\mathbb{Q}$. For $p \notin \Omega_0$, let $G_{(p)} := \mathrm{GSpin}(L \otimes \mathbb{Z}_{(p)}, Q)$, which is a reductive model for G over $\mathbb{Z}_{(p)}$ and naturally a subgroup of $C(L\otimes\mathbb{Z}_{(p)})^\times$, where $C(-)$ denotes the Clifford algebra. The group $G(\mathbb{R})$ acts on the Hermitian symmetric domain $D_V = \{z \in V_\mathbb{C} \mid [z, z] = 0, [z, \bar{z}] < 0\}/\mathbb{C}^\times$ via $G_\mathbb{Q} \to \mathrm{SO}(V)$. For $[z] \in D_V$ with $z \in V_\mathbb{C}$, let $h_{[z]} : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\, \mathbb{G}_m \to G_\mathbb{R}$ denote the unique homomorphism which induces the Hodge decomposition on $V_\mathbb{C}$ given by

$$V_\mathbb{C}^{1,-1} = \mathbb{C}z, V_\mathbb{C}^{0,0} = (\mathbb{C}z \oplus \mathbb{C}\bar{z})^\perp, V_\mathbb{C}^{-1,1} = \mathbb{C}\bar{z}.$$

Thus (G, D_V) is a Shimura datum with reflex field $\mathbb{Q}$.

Let $\mathbb{K} \subset G(\mathbb{A}_f)$ be a compact open subgroup contained in $\mathbb{K}_L := G(\mathbb{A}_f) \cap C(L \otimes \widehat{\mathbb{Z}})^\times$, where $C(L \otimes \widehat{\mathbb{Z}})$ denotes the Clifford algebra of $(L \otimes \widehat{\mathbb{Z}}, Q)$. Then we have the GSpin Shimura variety $S(G_\mathbb{Q}, D_V)_\mathbb{K}$ over $\mathbb{Q}$ with $S(G_\mathbb{Q}, D_V)_\mathbb{K}(\mathbb{C}) = G(\mathbb{Q})\backslash D_V \times G(\mathbb{A}_f)/\mathbb{K}$. If $p \notin \Omega_0$ and $\mathbb{K}_p = G_{(p)}(\mathbb{Z}_p)$, then by [Kis10, Theorem 2.3.8], $S(G_\mathbb{Q}, D_V)_\mathbb{K}$ admits a canonical smooth integral model $\mathcal{S}_{(p)}$ over $\mathbb{Z}_{(p)}$. Moreover, [AGHMP18, Theorem 4.4.6] constructed a flat normal Deligne–Mumford $\mathbb{Z}_\Omega$-stack $\mathcal{S}$ when $\mathbb{K} = \mathbb{K}_L$, which coincides with the canonical smooth integral model $\mathcal{S}_{(p)}$ at $p \notin \Omega_0$. If $\mathbb{K} \neq \mathbb{K}_L$, for the rest of the paper, we always enlarge Ω_0 and Ω to include all primes ℓ such that $\mathbb{K}_\ell \neq \mathbb{K}_{L,\ell}$. Henceforth, we will use the terminology $\mathcal{S}_\mathbb{Q}$ to denote $S(G_\mathbb{Q}, D_V)_\mathbb{K}$.

3.2. Let H denote the Clifford algebra $C(L)$ equipped with the right action by itself via right multiplication. Moreover, $H \otimes \mathbb{Q}$ is equipped with the action of G by left multiplication. By picking a suitable symplectic form on H, we have $G \to \mathrm{GSp}(H\otimes\mathbb{Q})$, which induces a morphism from (G, D_V) to a Siegel Shimura datum. Thus there is a Kuga–Satake abelian scheme $A^{\mathrm{univ}} \to \mathcal{S}_\mathbb{Q}$ whose first $\mathbb{Z}$-coefficient Betti cohomology $\mathbf{H}_B$ is the local system induced by H (and its G-action). This Kuga–Satake abelian scheme $A^{\mathrm{univ}} \to \mathcal{S}_\mathbb{Q}$ extends to an abelian scheme $\mathcal{A}^{\mathrm{univ}} \to \mathcal{S}$ equipped with a left $C(L)$-action by [AGHMP18, Theorem 4.4.6]. Let $\mathbf{H}_{\mathrm{dR}}, \mathbf{H}_{\ell,\text{ét}}$ denote the first relative de Rham cohomology of $\mathcal{A}^{\mathrm{univ}} \to \mathcal{S}$ and ℓ-adic étale cohomology with $\mathbb{Z}_\ell$-coefficient of $\mathcal{A}^{\mathrm{univ}}_{\mathbb{Z}_\Omega[1/\ell]} \to \mathcal{S}_{\mathbb{Z}_\Omega[1/\ell]}$, and let $\mathbf{H}_{\mathrm{cris}}$ denote the first relative crystalline cohomology of $\mathcal{A}^{\mathrm{univ}}_{\mathbb{F}_p} \to \mathcal{S}_{\mathbb{F}_p}$ for $p \notin \Omega_0$.

The action of L on H via left multiplication induces a G-equivariant map $L\otimes\mathbb{Q} \to \mathrm{End}_{C(L)}(H \otimes \mathbb{Q})$, and thus we have a $\mathbb{Z}$-local system $\mathbf{L}_B$ over $\mathcal{S}_\mathbb{Q}$ with a natural embedding $\mathbf{L}_B \to \mathrm{End}_{C(L)}(\mathbf{H}_B)$. There are a $\mathbb{Z}_\ell$-lisse sheaf $\mathbf{L}_{\ell,\text{ét}} \subset \mathrm{End}_{C(L)}(\mathbf{H}_{\ell,\text{ét}})$ over $\mathbb{Z}_\Omega[1/\ell]$, a filtered vector bundle with connection $\mathbf{L}_{\mathrm{dR}} \subset \mathrm{End}_{C(L)}(\mathbf{H}_{\mathrm{dR}})$ over $\mathbb{Z}_{\Omega_0}$, and an F-crystal $\mathbf{L}_{\mathrm{cris}} \subset \mathrm{End}_{C(L)}(\mathbf{H}_{\mathrm{cris}})$ for $p \notin \Omega_0$, such that these embeddings along with $\mathbf{L}_B \to \mathrm{End}_{C(L)}(\mathbf{H}_B)$ are compatible under Betti-de Rham, Betti-étale, de Rham-crystalline comparison maps (see [MP16, Prop. 3.11, 3.12, Prop. 4.7]). By [AGHMP18, §4.3], $\mathbf{L}_?, ? = B, \mathrm{dR}, (\ell, \text{ét}), \mathrm{cris}$ are equipped with a natural quadratic form $\mathbf{Q}$ given by $f \circ f = \mathbf{Q}(f) \cdot \mathrm{Id}$ for a section f of $\mathbf{L}_?$.

Definition 3.3 ([AGHMP18, Def. 4.3.1, §4.5]; [HP20, §6.4])**.** Let T denote an $\mathcal{S}$-scheme.
If T is an $\mathcal{S}_{\mathbb{Z}_{\Omega_0}}$-scheme,

1. An endomorphism $\lambda \in \mathrm{End}_{C(L)}(\mathcal{A}^{\mathrm{univ}}_T)$ is *special* if all cohomological realizations of λ lie in the image of $\mathbf{L}_? \to \mathrm{End}_{C(L)}(\mathbf{H}_?)$, where $? = B, \mathrm{dR}, \mathrm{cris}, (\ell, \text{ét})$.[7]

2. Assume that $T \otimes \mathbb{F}_p \neq \emptyset$. Let $\mathcal{A}^{\mathrm{univ}}_T[p^\infty]$ denote the p-divisible group associated to $\mathcal{A}^{\mathrm{univ}}_T$. An endomorphism $\lambda \in \mathrm{End}_{C(L)}(\mathcal{A}^{\mathrm{univ}}_T[p^\infty])$ is *special* if its crystalline realization lies in $\mathbf{L}_{\mathrm{cris}}$.

[7]We drop the ones which do not make sense.

In general, we choose for each $p \in \Omega_0 \setminus \Omega$, an embedding of L into a lattice $L^\diamond$ self-dual at p (and denote the universal family by $\mathcal{A}^{\diamond,\mathrm{univ}}$) and define *special endomorphisms* λ in the same way as above using $\mathbf{L}_?$, where $? = B, \mathrm{dR}, (\ell, \text{ét}), \mathrm{cris}$ for $p \notin \Omega_0$, and for $p \in \Omega_0 \setminus \Omega$, we observe that naturally $\mathrm{End}_{C(L)}(\mathcal{A}_T^{\mathrm{univ}}) \subset \mathrm{End}_{C(L^\diamond)}(\mathcal{A}_T^{\diamond,\mathrm{univ}})$ and further assume that λ lies in the image of $\mathbf{L}_?^\diamond$. When $T \otimes \mathbb{F}_p \neq \emptyset$ and $p \in \Omega_0 \setminus \Omega$, An endomorphism $\lambda \in \mathrm{End}_{C(L)}(\mathcal{A}_T^{\mathrm{univ}}[p^\infty])$ is *special* if its crystalline realization lies in $\mathbf{L}_{\mathrm{cris}}^\diamond$. Both definitions are independent of the choice of $L^\diamond$.

Remark 3.4. By [MP16, Lem. 5.2], for $\lambda \in \mathrm{End}_{C(L)}(\mathcal{A}_T^{\mathrm{univ}})$ special, we have $\lambda \circ \lambda = [Q(\lambda)]$ for some $Q(\lambda) \in \mathbb{Z}_{\geq 0}$ and $\lambda \mapsto Q(\lambda)$ is a positive definite quadratic form on the $\mathbb{Z}$-lattice of special endomorphisms of $\mathcal{A}_T^{\mathrm{univ}}$.

Definition 3.5. For $m \in \mathbb{Z}_{>0}$, the *special divisor* $\mathcal{Z}(m)$ is the Deligne–Mumford stack over $\mathcal{S}$ with functor of points $\mathcal{Z}(m)(T) = \{\lambda \in \mathrm{End}(\mathcal{A}_T^{\mathrm{univ}}) \text{ special } | Q(\lambda) = m\}$ for any $\mathcal{S}$-scheme T. We use the same notation for the image of $\mathcal{Z}(m)$ in $\mathcal{S}$. By [HP20, Prop. 6.5.2], $\mathcal{Z}(m)$ is étale locally an effective Cartier divisor. For $p > 2$ and $b \geq 3$ or for $p \notin \Omega_0$, the divisor $\mathcal{Z}(m)$ is flat over $\mathbb{Z}_{(p)}$ and hence $\mathcal{Z}(m)_{\mathbb{F}_p}$ is still an effective Cartier divisor of $\mathcal{S}_{\mathbb{F}_p}$. We denote $\mathcal{Z}(m)_{\mathbb{Q}}$ by $Z(m)$.

3.6. If (V, Q) is anisotropic (this implies $b \leq 2$), then by [MP19, Cor. 4.1.7], $\mathcal{S}$ is a proper Deligne–Mumford stack over $\mathbb{Z}_\Omega$. For the rest of this paragraph, we work with (V, Q) isotropic and recall the description of toroidal compactifications of $\mathcal{S}$. The construction of the integral model of toroidal compactions is to glue together models for all primes p. Fix a prime p, we choose a suitable embedding of the Shimura datum (G, D_V) to a Siegel Shimura datum[8] $(G^{\mathrm{Sg}}, D^{\mathrm{Sg}})$ and shrink the prime-to-p level to make $\mathbb{K}$ and the level $\mathbb{K}^{\mathrm{Sg}}$ of the Siegel variety both neat. We choose a smooth $\mathbb{K}^{\mathrm{Sg}}$-admissible complete finite rational polyhedral cone decomposition Σ^{Sg} of $(G^{\mathrm{Sg}}, D^{\mathrm{Sg}})$ satisfying the no self-intersection property and it induces a smooth $\mathbb{K}$-admissible complete finite rational polyhedral cone decomposition Σ of (G, D_V) also satisfying the no self-intersection property.[9] To glue together, we take a common refinement of all the above Σ (we only need finitely many to cover all p) and still denote it by Σ. By [MP19, 4.1.4, Thm. 4.1.5, Rmk. 4.1.6], the proper normal Deligne–Mumford $\mathbb{Z}_\Omega$-stack $\mathcal{S}^\Sigma$ obtained as glueing together the $\mathbb{Z}_{(p)}$-models constructed as the normalization of the Faltings–Chai integral compactification [FC90] of $\mathcal{S}(G^{\mathrm{Sg}}, D^{\mathrm{Sg}})_{\mathbb{K}^{\mathrm{Sg}}}$ in $\mathcal{S}_{\mathbb{Q}}$ is a compactification of $\mathcal{S}$ extending the toroidal compactification $\mathcal{S}_{\mathbb{Q}}^\Sigma$ of $\mathcal{S}_{\mathbb{Q}}$ with respect to Σ over $\mathbb{Q}$ constructed in [AMRT10, Pin90]. The integral compactification $\mathcal{S}^\Sigma$ depends on the choice of Σ but is independent of the choice of Hodge embedding $(G^{\mathrm{Sg}}, D^{\mathrm{Sg}})$, and the choice of cone decomposition Σ^{Sg}. The boundary $\mathcal{S}^\Sigma \setminus \mathcal{S}$ is a Cartier divisor. Further, $\mathcal{S}^\Sigma$ admits a stratification by locally closed substacks indexed by $\mathbb{K}$-isomorphism classes of toroidal stratum representatives (see [HP20, Def. 2.4.5, Def. 2.4.6] or [MP19, 2.1.26]). Each stratum is flat over $\mathbb{Z}_\Omega$ and is given by quotient of integral model of certain mixed Shimura variety by a finite group. The formal completion of $\mathcal{S}^\Sigma$ along each boundary component is of the same shape as that of $\mathcal{S}_{\mathbb{Q}}^\Sigma$. In short, as long as we work with a sufficiently fine cone decomposition Σ, the integral toroidal compactification $\mathcal{S}^\Sigma$ exists and has the stratification properties listed above.

Howard–Pappas [HP20, Prop. 8.1.2] provide the extension of automorphic vector bundles to the integral toroidal compactification in the hyperspecial case. More precisely, given an algebraic

[8] Using previous notation, we may choose $G^{\mathrm{Sg}} = \mathrm{GSp}(C(L^\diamond))$.

[9] See for instance [HP20, §2.4], [MP19, §§2.1.17-2.1.23, 2.1.28] for the definition and see [MP19, Thm. 2.1.25], [HP20, Remarks 2.4.9, 2.6.1] and the references cited in *loc. cit.* for the existence of such cone decomposition Σ^{Sg} after maybe further shrinking the prime-to-p part of $\mathbb{K}^{\mathrm{Sg}}$. We also remark that these properties of cone decompositions still hold if one needs to further refine Σ and Σ^{Sg}.

representation of G on a $\mathbb{Q}$-vector space N, consider the corresponding vector bundle with filtration on $\mathcal{S}_{\mathbb{Q}}$. Work of Harris and Harris–Zucker (see for instance [HZ01, §1]) constructs a canonical extension of the vector bundle with filtration to $\mathcal{S}_{\mathbb{Q}}^{\Sigma}$ functorial in N. This is summarized in [HP20, §3, Thm. 3.4.1]. By [HP20, Prop. 8.1.2], this canonical extension on $\mathcal{S}_{\mathbb{Q}}^{\Sigma}$ extends functorially to the integral model $\mathcal{S}_{\mathbb{Z}_{\Omega_0}}^{\Sigma}$. In particular, these results provide a canonical extension of the Hodge line bundle on $\mathcal{S}_{\mathbb{Q}}$ to $\mathcal{S}_{\mathbb{Z}_{\Omega_0}}^{\Sigma}$. In order to define the Hodge line bundle $\boldsymbol{\omega}$ on $\mathcal{S}^{\Sigma}$, for each $p \in \Omega_0 \setminus \Omega$, [HP20] chose auxiliary self-dual lattices $L^{\diamond} \supset L$ at p, and define the Hodge line bundle on $\mathcal{S}^{\Sigma}$ to be the one obtained by pulling back the Hodge line bundle on a suitable integral model of toroidal compactification of the GSpin Shimura variety associated to $L^{\diamond}$.

We will abuse notation and also denote the Zariski closure of the special divisor $\mathcal{Z}(m)$ in $\mathcal{S}^{\Sigma}$ by $\mathcal{Z}(m)$.

3.7. We now recall various results on modularity of the generating series of special divisors. These results will be used in understanding the (arithmetic) intersections of an arithmetic 1-cycle with special divisors, especially when $\dim \mathcal{S} \geq 3$.

For simplicity, we first recall the theorem by Howard and Madapusi on the generating series of special divisors in $\mathcal{S}$. Since the pullback of the Hodge line bundle $\boldsymbol{\omega}$ to the Hermitian symmetric domain D_V is the tautological line bundle, they endow $\boldsymbol{\omega}$ with the metric $||z||^2 = -\dfrac{[z, \bar{z}]}{4\pi e^{-\Gamma'(1)}}$ and denote this metrized line bundle by $\widehat{\boldsymbol{\omega}}$. They also endow the special divisor $\mathcal{Z}(m)$ with a Green function Φ_m defined by Bruinier [Bru02, (2.16) and the paragraphs above and below]. (More precisely, we take $s = 1/2 + b/4, \beta = 0$ in the expression $\Phi_{\beta,m}(v, s)$ defined in [Bru02, (2.16) and the paragraph above].[10] In *loc. cit.*, Bruinier follows work of Borcherds [Bor98] and defines the the regularized theta lifting of the Hejhal–Poincaré harmonic Maass form F_m of weight $1 - b/2$ with respect to the dual of the unitary Weil representation whose principal part is $\mathfrak{e}_0 q^{-m}$. Here $\{\mathfrak{e}_\beta\}_{\beta \in L^\vee/L}$ denotes the standard basis of $\mathbb{C}[L^\vee/L]$.)[11] Let $\widehat{\mathcal{Z}}(m)$ denote the arithmetic divisor $(\mathcal{Z}(m), \Phi_m)$ on $\mathcal{S}$. Since $\mathcal{Z}(m)$ is étale locally a Cartier divisor and $\mathcal{S}$ is normal, we have that the first arithmetic Chow group (of $\mathbb{Q}$-divisors) $\widehat{\mathrm{CH}}^1(\mathcal{S})_{\mathbb{Q}}$ is isomorphic to the group of isomorphism classes of $\mathbb{Q}$-line bundles $\widehat{\mathrm{Pic}}(\mathcal{S})_{\mathbb{Q}}$ and then we may view $\widehat{\mathcal{Z}}(m)$ and $\widehat{\boldsymbol{\omega}}$ as elements in $\widehat{\mathrm{CH}}^1(\mathcal{S})_{\mathbb{Q}}$. (See for instance [AGHMP17, §5.1] for details.)

By [HP20, Thm. 9.5.1], when $b \geq 3$, the generating series

$$-\widehat{\boldsymbol{\omega}} + \sum_{m \in \mathbb{Z}_{>0}} \widehat{\mathcal{Z}}(m) q^m$$

is a weight $1 + \frac{b}{2}$ modular form with coefficients in $\widehat{\mathrm{CH}}^1(\mathcal{S})_{\mathbb{Q}}$.[12] More precisely, for any $\mathbb{Q}$-linear

[10] Here we mean exactly the regularized theta lifting defined using [Bru02, (2.16)]; in the rest of [Bru02, §2], the notation $\Phi_{\beta,m}(v, s)$ is used to denote a slightly different regularization, which is related to ϕ_m in our notation defined in §4.3.

[11] The terms used in (2.16) in *loc. cit.* are defined in Def. 1.8, eqns (2.2), (2.14) — see [BY09, §§3.1, 4] for a summary. See also [AGHMP17, §3.2], [SSTT22, §3.1], and references therein.

[12] More precisely, it is one component of a vector valued modular form with respect to the unitary Weil representation of the metaplectic double cover $\mathrm{Mp}_2(\mathbb{Z})$ of $\mathrm{SL}_2(\mathbb{Z})$. The assumption $b \geq 3$ is harmless for us as the proof of our main results for $b \leq 2$ does not need such modularity result. On the other hand, there are modularity results when $b \leq 2$; see for instance [BBGK07].

map $\alpha : \widehat{\mathrm{CH}}^1(\mathcal{S})_{\mathbb{Q}} \to \mathbb{C}$, we have

$$-\alpha(\widehat{\boldsymbol{\omega}}) + \sum_{m \in \mathbb{Z}_{>0}} \alpha(\widehat{\mathcal{Z}}(m)) q^m$$

is a weight $1 + \frac{b}{2}$ modular form.

The $b \geq 3$ assumption above was only used in the treatment of the Green functions. For $\mathcal{Z}(m)$ and $\boldsymbol{\omega}$ in $\mathrm{CH}^1(\mathcal{S})_{\mathbb{Q}}$, Howard and Madapusi [HP20, Thm. 9.4.1] proved that for arbitrary b, the generating series

$$-\boldsymbol{\omega} + \sum_{m \in \mathbb{Z}_{>0}} \mathcal{Z}(m) q^m$$

is a weight $1 + \frac{b}{2}$ modular form with coefficients in $\mathrm{CH}^1(\mathcal{S})_{\mathbb{Q}}$. Thus for $p \notin \Omega_0$, since all $\mathcal{Z}(m)$ are flat over $\mathbb{Z}_{(p)}$, by intersecting the above generating series with $\mathcal{S}_{\mathbb{F}_p}$, we have

$$-\boldsymbol{\omega}_{\mathbb{F}_p} + \sum_{m \in \mathbb{Z}_{>0}} \mathcal{Z}(m)_{\mathbb{F}_p} q^m$$

is a weight $1 + \frac{b}{2}$ modular form with coefficients in $\mathrm{CH}^1(\mathcal{S}_{\mathbb{F}_p})_{\mathbb{Q}}$.

We now recall the modularity result for toroidal compactification $\mathcal{S}^\Sigma$. Since the natural metrics on $\widehat{\boldsymbol{\omega}}$ and Green functions Φ_m (even after suitable adjustments due to correction terms using boundary divisors) have singularities along the boundary, we need to work with the first arithmetic Chow ring with pre-log-log forms $\widehat{\mathrm{CH}}^1(\mathcal{S}^\Sigma, \mathcal{D}_{\mathrm{pre}})$ introduced by Burgos Gil, Kramer, and Kühn in [BGKK07, §7] (see [BBGK07, §1] for a detailed summary). In short, we will work with $\widehat{\mathrm{CH}}^1(\mathcal{S}^\Sigma, \mathcal{D}_{\mathrm{pre}})_{\mathbb{Q}} := \widehat{\mathrm{CH}}^1(\mathcal{S}^\Sigma, \mathcal{D}_{\mathrm{pre}}) \otimes \mathbb{Q}$ (see [BBGK07, §1.4, Def. 1.15, §1.2] for the definition), the group of the isomorphism classes of $\mathbb{Q}$-Cartier divisors with Green functions satisfying certain growth condition along the boundary (see [BBGK07, Defs. 1.2, 1.3]).[13] Moreover, by [BBGK07, pp. 20-21], $\widehat{\mathrm{CH}}^1(\mathcal{S}^\Sigma, \mathcal{D}_{\mathrm{pre}})_{\mathbb{Q}}$ is isomorphic to the arithmetic Picard group of isomorphic classes of $\mathbb{Q}$-Hermitian line bundles allowing pre-log singularity of the Hermitian metric along the boundary.

Bruinier and Zemel [BZ21, Thms. 1.2, 4.19, Cor. 4.15] analyzed the behavior of Φ_m along boundary divisors and proved that after introducing suitable correction terms to $Z(m)$ from boundary divisors, Φ_m is pre-log-log along the boundary and the generating series of special divisors (with correction terms from boundary divisors) is modular in $\mathrm{CH}^1(\mathcal{S}^\Sigma_{\mathbb{Q}})_{\mathbb{Q}}$.[14] They also remarked [BZ21, Rmk. 5.6] that with their correction terms, they also obtain the modularity of generating series of special divisors in $\widehat{\mathrm{CH}}^1(\mathcal{S}^\Sigma, \mathcal{D}_{\mathrm{pre}})_{\mathbb{Q}}$. See Tayou's paper [Tay24, Thm. 3.1] for a detailed discussion using inputs from the proof of the main theorem in [HP20, §9].

We now recall the precise statement of the modularity result following [BZ21] for $b \geq 3$. [BZ21, §§3.2, 3.3] provides an explicit description of $\mathcal{S}^\Sigma_{\mathbb{C}}$ and [BZ21, §§4.2-4.4] provides explicit formulae for the coefficients of the correction terms arising from boundary divisors. Based on the summary

[13] Although in [BBGK07] they work with regular scheme, their whole discussion on the Archimedean places and Green functions apply without changes to our normal stack $\mathcal{S}^\Sigma$ once we restrict ourselves to $\mathbb{Q}$-Cartier divisors.

[14] Strictly speaking, Bruinier and Zemel did not prove that the coefficients of the correction terms from the boundary divisors are rational and thus only obtained their result in $\mathrm{CH}^1(\mathcal{S}^\Sigma_{\mathbb{Q}})_{\mathbb{R}}$. The rationality of coefficients is later proved by Engel, Greer and Tayou [EGT]. In their paper, they also provide an alternative proof of the modularity of generating series.

above, these results in [BZ21] hold without change to $\mathcal{S}^\Sigma$ and $\mathcal{S}^\Sigma_{\mathbb{F}_p}$. See also [Tay24, §§2.2.1, 2.2.2, 4.5, 4.6].

Recall that $\mathcal{S}^\Sigma$ admits a stratification indexed by $\mathbb{K}$-isomorphism classes of toroidal stratum representatives and thus the irreducible components in the boundary can be grouped in terms of equivalence classes the corresponding cusp label representatives.[15] Since $\mathcal{S}_\mathbb{Q}$ is a GSpin Shimura variety, $\mathcal{S}^\Sigma$ has the following two types of cusp label representatives: the admissible parabolic subgroup P of G in a cusp label representative is the stabilizer of either an isotropic plane $J_\mathbb{Q} \subset V$ or an isotropic line $I_\mathbb{Q} \subset V$.

When P in the cusp label representative Φ is the stabilizer of an isotropic plane $J_\mathbb{Q}$, then the boundary component in $\mathcal{S}^\Sigma$ above this cusp is canonical.[16] The multiplicity $\mathrm{mult}_{J_\mathbb{Q}}(m)$ of the boundary divisor (being the Zariski closure in $\mathcal{S}^\Sigma$ of the stratum associated to Φ) in the correction term to $\mathcal{Z}(m)$ is given by

$$\mathrm{mult}_{J_\mathbb{Q}}(m) := \frac{m}{b-2}|\{v \in J_L^\perp/J \mid Q(v) = m\}|,$$

where $J := J_\mathbb{Q} \cap L, J_L^\perp := J_\mathbb{Q}^\perp \cap L$. Thus $J_L^\perp/J$ is equipped with a quadratic form induced from Q on V and it is a positive definite lattice (see [BZ21, Prop. 4.21, eqn. (4.28)]).[17]

When P in Φ is the stabilizer of an isotropic line $I_\mathbb{Q}$, the boundary components in $\mathcal{S}^\Sigma$ above the cusp Φ depend on the choice of the cone decomposition. More precisely, the irreducible boundary divisors above a given Φ are indexed by $\mathbb{K}$-isomorphism classes of rational rays $\mathbb{R}_{>0}w$ in the open convex cone C_Φ (defined in [HP20, (2.4.1)]). By [BZ21, pp. 19-20], C_Φ is a connected component of $\{v \in (I_\mathbb{Q}^\perp/I_\mathbb{Q}) \otimes_\mathbb{Q} \mathbb{R} \mid Q(v) < 0\}$ (note that Q naturally induces a quadratic form on $(I_\mathbb{Q}^\perp/I_\mathbb{Q}) \otimes_\mathbb{Q} \mathbb{R}$).

The multiplicity $\mathrm{mult}_{I_\mathbb{Q},\mathbb{R}_{>0}w}$ of the boundary divisor in the correction term to $\mathcal{Z}(m)$ is given by

$$\mathrm{mult}_{I_\mathbb{Q},\mathbb{R}_{>0}w}(m) := \frac{\sqrt{-Q(w)}}{8\sqrt{2}\pi}\Phi_m^K\left(\frac{w}{\sqrt{-Q(w)}}\right),$$

where $K := (I_\mathbb{Q}^\perp \cap L)/(I_\mathbb{Q} \cap L)$ and we choose w in the given ray $\mathbb{R}_{>0}w$ to be the primitive element in $K \cap C_\Phi$. The function Φ_m^K denotes the regularized theta lifting (using the Siegel theta function associated to the lattice K) of the Poincaré series on K obtained naturally from F_m (see [BZ21, Def. 4.18, eqn. (4.8)] — we omit their $p_K\mu$ in our notation as we always work with $\mu = 0$. The discussions on Poincaré series and theta liftings in [Bru02, §§1.3-2.3] include the case of lattices of signature $(b-1, 1)$ and thus apply to K here).

Note that the multiplicity only depends on $J_\mathbb{Q}$ or $(I_\mathbb{Q}, \mathbb{R}_{>0}w)$ in either case. Given an isotropic plane $J_\mathbb{Q}$, let $\mathcal{B}_{J_\mathbb{Q}}$ denote the boundary divisor consisting of all irreducible boundary divisors in $\mathcal{S}^\Sigma$ above the cusp label representatives which are $\mathbb{K}$-isomorphic to a cusp label representative whose admissible parabolic group is the stabilizer of $J_\mathbb{Q}$. Given a pair $(I_\mathbb{Q}, \mathbb{R}_{>0}w)$ of an isotropic line and a rational ray in its open convex cone, let $\mathcal{B}_{I_\mathbb{Q},\mathbb{R}_{>0}w}$ denote the boundary divisor consisting of all irreducible boundary divisors in $\mathcal{S}^\Sigma$ whose toroidal stratum representatives are $\mathbb{K}$-isomorphic to a

[15] Geometrically speaking, the boundary components or cusps of the Baily–Borel compactification of $\mathcal{S}$ are indexed by the set of equivalence classes of cusp label representatives [MP19, §5.2.4, Thm. 5.2.11]; here we consider the boundary divisors in $\mathcal{S}^\Sigma$ lying above a given cusp.

[16] Indeed, the boundary components above a given cusp depending on the choice of the cone decomposition in C_Φ^* defined on [HP20, p. 201] and in this case, following for instance the discussions and notation in [HP20, §§2.4, 4.5], we have $C_\Phi^* = \mathbb{R}_{\geq 0}$ and the only possible cone decomposition is $C_\Phi^* = \{0\} \cup \mathbb{R}_{>0}$.

[17] Note that there is a slight difference in notation —- our m is the same as $2m$ in [BZ21].

toroidal stratum representative whose admissible parabolic group is the stabilizer of $I_{\mathbb{Q}}$ and the rational polyhedral cone is $\mathbb{R}_{>0}w$. Define

$$\mathcal{Z}^{\Sigma}(m) = \mathcal{Z}(m) + \sum_{J_{\mathbb{Q}}} \mathrm{mult}_{J_{\mathbb{Q}}}(m)\mathcal{B}_{J_{\mathbb{Q}}} + \sum_{I_{\mathbb{Q}},\mathbb{R}_{>0}w} \mathrm{mult}_{I_{\mathbb{Q}},\mathbb{R}_{>0}w}\, \mathcal{B}_{I_{\mathbb{Q}},\mathbb{R}_{>0}w},$$

where $J_{\mathbb{Q}}$ and $(I_{\mathbb{Q}}, \mathbb{R}_{>0}w)$ run through a set of representatives of $\mathbb{K}$-isomorphism classes. Then $\widehat{\mathcal{Z}}^{\Sigma}(m) := (\mathcal{Z}^{\Sigma}(m), \Phi_m)$ is an arithmetic divisor (with pre-log-log form).

[BZ21, Thm. 1.2] and [Tay24, Thm. 3.1] show that the generating series

$$-\widehat{\boldsymbol{\omega}} + \sum_{m\in\mathbb{Z}_{>0}} \widehat{\mathcal{Z}}^{\Sigma}(m)q^m$$

is a weight $1 + \frac{b}{2}$ modular form with coefficients in $\widehat{\mathrm{CH}}^1(\mathcal{S}^{\Sigma}, \mathcal{D}_{\mathrm{pre}})_{\mathbb{Q}}$.

There are also other related works in the literature. Bruinier, Howard, Kudla, Rapoport and Yang [BHK+20] consider the case of Shimura varieties associated to a Hermitian space over an imaginary quadratic field of signature $(n, 1)$. They establish the modularity of generating series of special divisors in the canonical toroidal compactification of the Krämer integral model. In the case when the Shimura variety has a regular proper integral model, Zhang's work [Zha22, §2.5] provides an alternative way to lift the special divisors on the generic fiber canonically to elements in the Arakelov Chow group of the integral model, and deduces the modularity of the generating series in the Chow group of the integral model from that of the generic fiber. Examples of such Shimura varieties are GSpin varieties associated to self dual quadratic forms over a totally real field $F \neq \mathbb{Q}$, where the quadratic form is indefinite at exactly one real place and has signature $(b, 2)$ at that place. For certain unitary Shimura varieties with signature $(n, 1), (n+1, 0), \ldots, (n+1, 0)$, Qiu [Qiu] provides an explicit description of the extension of the special divisors on the generic fiber to integral model following Zhang's theory and proves a modularity result in this setting.

4 The main theorems and the ideas of proofs

4.1 Statements of the main theorems

Loosely speaking, the overarching result that we prove can be stated as follows. Let $\mathcal{S}$ be the integral model of a GSpin Shimura variety recalled in §3.1, and consider a normal arithmetic curve C that admits a finite map to $\mathcal{S}$. We will assume that the image of this curve is not contained in $\mathcal{Z}(m)$ for every $m \in \mathbb{Z}_{>0}$. We will work in two settings — the number field setting and the characteristic p setting. In the number field setting, we will consider $x \in \mathcal{S}(K)$ (where K is a number field), which induces a map $C = \operatorname{Spec} \mathcal{O}_K[\frac{1}{N}] \to \mathcal{S}$ (for a large enough integer N). In the characteristic p setting, we will work with a smooth quasi-projective curve $C/\overline{\mathbb{F}}_p$ and a finite map $C \to \mathcal{S}_{\overline{\mathbb{F}}_p}$. In either case, we will prove that infinitely many closed points of C are contained in $\bigcup_{m\in\mathbb{Z}_{>0}} \mathcal{Z}(m)$ using a local-global argument that goes back to work of Chai–Oort [CO06] and Charles [Cha18]. We will describe the overarching strategy and previous work [CO06, Cha18] in §4.2. As we will see from the statements of the main theorems, results of this sort have immediate consequences to the splitting of abelian surfaces and Picard ranks of K3 surfaces. This is because appropriate special divisors inside the moduli spaces of polarized abelian surfaces (resp. polarized K3 surfaces) parameterize split abelian surfaces (K3 surfaces with extra line bundles). We now state the main theorems.

Theorem 4.1 ([ST20, Thm. 1]). *Let K denote a number field, and suppose that A/K is an abelian surface admitting real multiplication by some real quadratic field. Then there are infinitely many non-Archimedean places v of K modulo which A is geometrically isogenous to the square of some elliptic curve.*

We remark that the above result falls into the intersection-theoretic framework mentioned above because the moduli space of abelian surfaces with real multiplication is a GSpin Shimura variety, and $\overline{\mathbb{F}}_p$-points of special divisors parameterize abelian surfaces isogenous to a self-product of elliptic curves.

As will be clear later on in the section, the splitting of abelian surfaces whose geometric endomorphism ring is $\mathbb{Z}$ requires significantly new ideas. Indeed, the local estimates needed to carry out the local-global strategy in the setting of Theorem 4.1 do not require the assumption that A is defined over a number field. However, such local estimate is not true in general — even in the case of abelian surfaces A with $\mathrm{End}(A_{\overline{K}}) = \mathbb{Z}$![18] Indeed, there exist abelian surfaces with supersingular reduction over local fields which are arbitrarily well approximated (in the v-adic metric) by CM abelian surfaces, and our local estimates using the ideas in the proof of Theorem 4.1 are simply not good enough unless we use the fact that A is defined over a number field. The new input is discussed in §4.6.2.

The characteristic p setting differs from the number field setting in the case of abelian surfaces. For instance, the most general analogue of Theorem 4.1 is false! There exist non-isotrivial abelian surfaces over global function fields which remain non-split modulo all but finitely many places.[19] However, these counterexamples (and their higher dimensional counterparts) all occur only outside the ordinary setting. As we will see, the generically ordinary versions of the number field results still hold. While establishing local estimates is more challenging (the equi-characteristic p deformation theory of abelian varieties and their endomorphisms is more complicated than the mixed characteristic theory), there are no non-isotrivial ordinary abelian surfaces over local function fields which are well-approximated by CM abelian surfaces. Therefore the local estimates we obtain do not require the assumption that the abelian surface is defined over a global function field. We obtain the following result.

Theorem 4.2 ([MST22, Thm. 1],[Tay24, Thm. 4.8][20]). *Let $p \geq 5$ be a prime. Let $C/\overline{\mathbb{F}}_p$ be a smooth irreducible quasi-projective curve with a finite non-constant morphism to $\mathcal{A}_{2,\overline{\mathbb{F}}_p}$ whose image is generically ordinary. Let $A/k(C)$ denote the abelian surface over the function field of C induced by this map.*

1. *Suppose that A has no extra endomorphisms. Then infinitely many $\overline{\mathbb{F}}_p$-points of C correspond to non-simple abelian surfaces.*
2. *Suppose that A admits real multiplication by some field whose discriminant is relatively prime to p. Then infinitely many $\overline{\mathbb{F}}_p$-points of C parameterize abelian surfaces isogenous to a self-product of an elliptic curve.*

[18] See Remark 4.10.

[19] See for instance [MST22, Rmk. 2].

[20] In order to obtain 1) in Theorem 4.2, we need to consider intersections with $\mathcal{Z}(m^2)$. Even though Tayou's theorem is only stated for $\mathcal{Z}(m)$, the only input we need is the local estimate at the bad reduction points on C given in Prop. 4.12 of *loc. cit.*. This bound is good enough to be combined with our proof in [MST22] to remove the projectiveness assumption on C.

As mentioned above, dealing with the higher dimensional case presents new challenges, which requires a global input. We will discuss this in detail in §4.6, and end this subsection by stating our main results. In the number field case, we prove the following result:

Theorem 4.3 ([SSTT22, Thm. 2.4],[Tay24, Thm. 4.1]). *Let $\mathcal{S}$ be a GSpin Shimura variety and let K be a number field. Let $x \in \mathcal{S}(K)$ denote a K-valued point of $\mathcal{S}$. Consider the $\mathcal{O}_{K,S}$-valued point of $\mathcal{S}$ (where S is a finite set of places of K) induced by x — we also denote the $\mathcal{O}_{K,S}$-valued point by x. Then there exist infinitely many places v of K such that $x \bmod v \in \bigcup_{m\in\mathbb{Z}_{>0}} \mathcal{Z}(m^2)$. Consequently:*

1. *Let X/K denote a K3 surface. Then there exist infinitely many places v of K such that the Picard rank of $X_{\overline{\mathbb{F}}_v}$ is strictly greater than the Picard rank of $X_{\overline{K}}$. Here $\overline{\mathbb{F}}_v$ denotes an algebraic closure of the residue field $\mathbb{F}_v$.*
2. *Let A/K denote an abelian surface. Then there exist infinitely many places v of K such $A_{\overline{\mathbb{F}}_v}$ is isogenous to a product of elliptic curves.*

We also have the following function field result:

Theorem 4.4 ([MST22b, Thm. 1.2],[Tay24, Thm. 4.8]). *Let $\mathcal{S}$ denote a GSpin Shimura variety associated to a quadratic form self-dual at p, where $p \geq 5$ is a prime. Let $C/\overline{\mathbb{F}}_p$ be a smooth irreducible quasi-projective curve with a finite non-constant morphism to $\mathcal{S}_{\overline{\mathbb{F}}_p}$ whose image is generically ordinary. Suppose further that the image of C is not contained in $\mathcal{Z}(m)_{\overline{\mathbb{F}}_p}$ for any $m \in \mathbb{Z}_{>0}$. Then infinitely many $\overline{\mathbb{F}}_p$-points of C map to $\bigcup_{m\in\mathbb{Z}_{>0}} \mathcal{Z}(m)_{\overline{\mathbb{F}}_p}$. Consequently, let $X/k(C)$ denote a non-isotrivial ordinary K3 surface and such that p does not divide the discriminant of the Picard lattice of X. Then infinitely many $\overline{\mathbb{F}}_p$-points of C parameterize K3 surfaces whose geometric Picard ranks are strictly greater than that of X.*

The original versions of Theorems 4.2, 4.3 and 4.4 had everywhere potentially good reduction hypotheses,[21] and were proved in [MST22], [SSTT22] and [MST22b] respectively. This good reduction hypothesis was lifted in work of Tayou [Tay24]. Indeed, Tayou's treatment of the boundary of $\mathcal{S}$ simultaneously handles the number field and characteristic p settings. We will discuss Tayou's work in §4.8. Theorem 4.4 has the following consequence:

Theorem 4.5 ([MST22b, Thm. 1.4]). *The ordinary Hecke orbit conjecture is true for $\mathcal{S}_{\mathbb{F}_p}$. Namely, let $x \in \mathcal{S}_{\mathbb{F}_p}(\overline{\mathbb{F}}_p)$ denote an ordinary point. Then the prime-to-p Hecke orbit of x is Zariski dense in $\mathcal{S}_{\mathbb{F}_p}$.*

In the following subsections, we will elaborate on the proofs of the above theorems, as well as the extra challenges that arise in characteristic p, and the crucial global inputs needed to prove Theorems 4.3 and 4.4.

4.2 Overarching strategy

Recall that C is a normal arithmetic curve which admits a finite map to $\mathcal{S}$ — i.e., $C = \operatorname{Spec} \mathcal{O}_K[1/N]$ (K is a number field), or C is an irreducible smooth curve over $\overline{\mathbb{F}}_p$. The basic strategy to prove the

[21] Work of Daniel Bragg and Ziquan Yang [BY23, Thm. 1.8 and the discussion below] shows that for $p \geq 3$ and a K3 surface X with a line bundle of degree prime to p, X has potentially good reduction at a place v above p if and only if the Kuga–Satake abelian variety associated to X has potentially good reduction at v.

results recalled in §4.1 is to define and estimate the intersection number $(C.\mathcal{Z}(m))$ as m varies, and to decompose this intersection number into a sum of local contributions $i_v(C.\mathcal{Z}(m))$, where $v \in C$ is a point.[22] The local contribution at v is positive precisely when $v \in \mathcal{Z}(m)$. The theorem would follow if one can prove that, as $m \to \infty$,

$$i_v(C.\mathcal{Z}(m)) = o(C.\mathcal{Z}(m)) \tag{4.2.1}$$

for any fixed point v of C. Indeed, the assertion (4.2.1) would imply that for any finite set S of points $v \in C$, $\sum_{v\in S} i_v(C.\mathcal{Z}(m)) < (C.\mathcal{Z}(m))$ for $m \gg_S 1$, which would then imply the existence of a new point $v \in C \setminus S$ with $v \in \mathcal{Z}(m)$ for some m.

As we will see later, in the number field version, we prove an average version of (4.2.1), averaged over points of C. An averaged version of (4.2.1) is not true if $C \subset \mathcal{S}_{\overline{\mathbb{F}}_p}$, but we establish strong enough local bounds and keep careful track of constants to prove our main results.

This strategy first appears in the setting of elliptic curves and the modular curve in work of Chai–Oort [CO06] (in the characteristic p setting) and work of Charles [Cha18] (in the number field case). We now elaborate on their work.

4.2.1 History: isogenies between elliptic curves

In [Cha18], Charles proves the following striking result.

Theorem 4.6 (Charles, [Cha18, Thm. 1.1])**.** *Let E_1, E_2 denote two elliptic curves over a number field K. Then there are infinitely many primes of K modulo which E_1 and E_2 are geometrically isogenous.*

The characteristic p analogue of Charles's theorem was proved earlier by Chai and Oort. Both approaches involve intersection theory on modular curves.

Theorem 4.7 (Chai–Oort, [CO06, Prop. 7.3])**.** *Let $C/\overline{\mathbb{F}}_p$ denote a quasi-projective curve, and suppose that $E_1, E_2/C$ are two non-isotrivial families of elliptic curves. Then there are infinitely many points $x \in C(\overline{\mathbb{F}}_p)$ such that $E_{1,x}$ is isogenous to $E_{2,x}$.*

Chai and Oort prove their result using intersection theory on $X(1) \times X(1)$, the moduli space of pairs of elliptic curves. Chai and Oort prove that C intersects Hecke-translates of the diagonal $\Delta \subset X(1) \times X(1)$ at infinitely many points of C, taken over an infinite family of Hecke correspondences. However, the family of correspondences they use are extremely particular to the setting of products of modular curves in positive characteristic — indeed the family they use is $\mathrm{Id} \times \mathrm{Frob}^n, n \in \mathbb{Z}_{>0}$, where Frob is the Frobenius on $X(1)$.

In the number field setting of this problem, there are not any analogues of these specific Hecke correspondences. Nevertheless, Charles considers the arithmetic intersection of j_1 and $T(j_2)$, where $j_i \in X(1)$ are the moduli points corresponding to (i.e. j-invariants of) the E_i and T varies over all Hecke correspondences on $X(1)$. More precisely, Charles endowed the divisor j_1 in $X(1)$ (over $\mathbb{Z}$) with a suitable Green function to make it into an Arakelov divisor $\widehat{j_1}$ and considers the Arakelov height of $T(j_2)$ with respect to j_1. This height is what we call the *global intersection number*. To uniformize the notation with the characteristic p setting, we use the term $(\widehat{j_1}, T(j_2))$ instead of using more standard height notation. Charles uses work of Autissier [Aut03] to estimate the

[22] Right now, we ignore that C may not be proper and extra places/points need to be considered. We will provide the precise definitions and details in §4.3.

global intersection number $(\widehat{j_1}, T(j_2))$. A key idea of Charles needed to bound the local intersection multiplicities is the following. If there exist two Hecke correspondences T, T' with the property that j_1 is v-adically close (in an appropriately quantified sense) to some elements of $T(j_2)$ and of $T'(j_2)$, then j_1 must be v-adically close to a CM point. He then uses the fact that CM points cannot be too close (again, in a suitably quantified sense) to each other, to obtain sufficient bounds on the local intersection multiplicities. We note that this case is analogous to the setting of abelian surfaces with real multiplication. Indeed, a product of modular curves $X(1) \times X(1)$ is a degenerate Hilbert modular surface, and Charles' setting can be viewed as studying the Arakelov height of Hecke translates of (j_1, j_2) with respect to the diagonal $X(1) \subset X(1) \times X(1)$. We refer the reader to §4.4 for more about the case of real multiplication.

4.3 Definitions of the intersection numbers

Let C denote a regular arithmetic curve mapping finitely to $\mathcal{S}$. We suppose that C is either a smooth quasi-projective curve over $\mathbb{F}_q$ (where q is a power of p), or that $C = \operatorname{Spec} \mathcal{O}_{K,S}$ where K is a number field and S is a finite set of primes. We assume that the generic point of C does not lie on any special divisor $\mathcal{Z}(m)$.

Let v be a non-Archimedean place of C, and let m denote a positive integer. Let $t \in \mathcal{O}_C$ denote a uniformizing parameter at v.

Definition 4.8. The *local intersection multiplicity* $i_v(C.\mathcal{Z}(m))$ is defined as[23]

$$i_v(C.\mathcal{Z}(m)) = \sum_{n=1}^{\infty} \#\{\lambda \in \operatorname{End}(\mathcal{A}^{\mathrm{univ}}_{\mathcal{O}_C/t^n}) \text{ special } \mid Q(\lambda) = m\} \tag{4.3.1}$$

Note that this sum is actually a finite sum because for $n \gg_m 1$, the set of $\lambda \in \operatorname{End}(\mathcal{A}^{\mathrm{univ}}_{\mathcal{O}_C/t^n})$ special and $Q(\lambda) = m$ is empty.

We would like to study

$$\sum_{v \in |C|} i_v(C.\mathcal{Z}(m)) \log |\mathbb{F}_v|,$$

where $\mathbb{F}_v$ is the residue field of the point/place v, as a global intersection number[24]. However this only has good intersection theoretic properties when C is a smooth proper curve (admitting a finite map to $\mathcal{S}$), in which case this sum is indeed the total intersection number $(C.\mathcal{Z}(m))$ in $\mathcal{S}$. Equivalently, as C must be a curve over $\mathbb{F}_q$ and the $\mathcal{Z}(m)$ we used in the proofs are flat over $\mathbb{Z}_p$, this number is (up to a factor of $\log q$) the classical intersection number $(C.\mathcal{Z}(m)_{\mathbb{F}_p})$ in $\mathcal{S}_{\mathbb{F}_p}$.

We now define what we mean by the global intersection number in all other settings.[25] If C is a not necessarily projective curve over $\mathbb{F}_q$, then the smooth projective compactification of C (which we will also denote by C) admits a finite morphism to any compactification of $\mathcal{S}$. We pick a suitable toroidal compactification $\mathcal{S}^{\Sigma}$ recalled in §3.6. Recall that we still use $\mathcal{Z}(m)$ to denote the

[23] This definition coincides with the usual local intersection multiplicity in both classical and arithmetic setting using the moduli interpretation of $\mathcal{Z}(m)$.

[24] In the proofs in [MST22, MST22b] in the characteristic p setting, we consider the sum over all $\overline{\mathbb{F}}_p$-points of C of the local intersection multiplicity $i_v(C.\mathcal{Z}(m))$ without the $\log |\mathbb{F}_v|$ term — these two definitions are equivalent up to a factor of $\log q$.

[25] In the following discussion, we will not spell out that one might need to pass to a finite cover of C because of stacky issues in order to have an extension to the compactification of $\mathcal{S}$.

Zariski closure of the special divisor in $\mathcal{S}^\Sigma$. Then the global intersection number to study in this case is $(C.\mathcal{Z}(m))$ in $\mathcal{S}^\Sigma$ (the same as $(C.\mathcal{Z}(m)_{\mathbb{F}_p})$ in $\mathcal{S}^\Sigma_{\mathbb{F}_p}$). Note that $(C.\mathcal{Z}(m))$ is the sum of local intersection multiplicities at all closed points of C. The local intersection multiplicities at points in $\mathcal{S}$ is given in (4.3.1), and the local intersection multiplicities at points in the boundary $\mathcal{S}^\Sigma \setminus \mathcal{S}$ have the usual intersection theoretic definition, which is sufficient for the purpose of this paper (see §4.8.2 for details).

If C is $\operatorname{Spec}\mathcal{O}_{K,S}$, then $\operatorname{Spec}\mathcal{O}_K$ (still denoted by C) admits a finite morphism to $\mathcal{S}^\Sigma$. The definition of the local intersection multiplicities at non-Archimedean places of K is the same as the characteristic p setting (i.e., (4.3.1) for primes of good reduction, and the intersection theoretic definition for primes modulo which C specializes to the boundary of $\mathcal{S}^\Sigma$).

We now discuss the suitable Archimedean term. For simplicity, we first consider the case when $\operatorname{Spec}\mathcal{O}_K$ lies in $\mathcal{S}$. As recalled in §3.7, on $\mathcal{S}$, each special divisor $\mathcal{Z}(m)$ is equipped with a Green function Φ_m and we may consider the Arakelov height $(C.(\mathcal{Z}(m),\Phi_m))$ of C with respect to $(\mathcal{Z}(m),\Phi_m)$ as the global intersection number. More precisely,

$$(C.(\mathcal{Z}(m),\Phi_m)) = \sum_{v \text{ non-Archimedean}} i_v(C.\mathcal{Z}(m)) \log|\mathbb{F}_v| + \sum_{\sigma: K\hookrightarrow\mathbb{C}} \Phi_m(C^\sigma),$$

where $i_v(C.\mathcal{Z}(m))$ is defined in (4.3.1) and C^σ denotes the $\mathbb{C}$-point on $\mathcal{S}$ obtained via $\sigma: K \hookrightarrow \mathbb{C}$. However, $\Phi_m(C^\sigma)$ grows much faster than both $(C.(\mathcal{Z}(m),\Phi_m))$ and $i_v(C.\mathcal{Z}(m))$ (for a fixed v on average over m) as $m\to\infty$ (see [SSTT22, Prop. 3.2, Thm. 5.7, Thm. 6.1, Thm. 7.1]). Therefore, to make the Archimedean term behave similar to the non-Archimedean term $i_v(C.\mathcal{Z}(m))$, we consider a different Green function. Following Bruinier's work [Bru02, Prop. 2.11], we write $\Phi_m = \phi_m - C_m$, where ϕ_m is the regularized theta lifting using Bruinier's regularization defined in [Bru02, (2.15) and the paragraph below] and C_m is a constant which is the derivative of a certain Fourier coefficient of Eisenstein series.[26] Note that ϕ_m is also a Green function for $\mathcal{Z}(m)$ in $\mathcal{S}$ and we have

$$(C.(\mathcal{Z}(m),\phi_m)) = \sum_{v \text{ non-Archimedean}} i_v(C.\mathcal{Z}(m)) \log|\mathbb{F}_v| + \sum_{\sigma: K\hookrightarrow\mathbb{C}} \phi_m(C^\sigma). \tag{4.3.2}$$

In the proofs of the number field setting, we refer to $(C.(\mathcal{Z}(m),\phi_m))$ as the global intersection number and $\phi_m(C^\sigma)$ as the Archimedean contribution. As we will see in the rest of this section, *on average over* m (away from a very small bad set of m) as $m\to\infty$, we have

$$i_v(C.\mathcal{Z}(m)) = o((C.(\mathcal{Z}(m),\phi_m))), \quad \phi_m(C^\sigma) = o((C.(\mathcal{Z}(m),\phi_m))). \tag{4.3.3}$$

In the general case $C = \operatorname{Spec}\mathcal{O}_K \to \mathcal{S}^\Sigma$, the above mentioned functions Φ_m, ϕ_m are Green functions (pre-log-log along the boundary) for

$$\mathcal{Z}(m)^\Sigma = \mathcal{Z}(m) + \sum_{J_\mathbb{Q}} \operatorname{mult}_{J_\mathbb{Q}}(m)\mathcal{B}_{J_\mathbb{Q}} + \sum_{I_\mathbb{Q},\mathbb{R}_{>0}w} \operatorname{mult}_{I_\mathbb{Q},\mathbb{R}_{>0}w}\mathcal{B}_{I_\mathbb{Q},\mathbb{R}_{>0}w}$$

[26] In [Bru02], he sometimes assumes that $b\geq 3$. We will not remark on which parts in [Bru02] hold true for $b \leq 2$ and will instead refer the reader to [BY09] as a reference for Green functions arising from regularized theta liftings. Moreover, we also remark that in the proofs of main theorems recalled in §4.1 for $b\leq 2$, one uses an alternative way to obtain Green functions with similar asymptotic on growth as ϕ_m discussed at the end of this subsection.

in §3.7. We have

$$(C.(\mathcal{Z}(m)^{\Sigma}, \phi_m)) = \sum_{v \text{ non-Archimedean}} i_v(C.\mathcal{Z}(m)) \log |\mathbb{F}_v| + \sum_{\sigma: K \hookrightarrow \mathbb{C}} \phi_m(C^{\sigma}) \tag{4.3.4}$$

$$+ \sum_{J_{\mathbb{Q}}} \mathrm{mult}_{J_{\mathbb{Q}}}(m)(C.\mathcal{B}_{J_{\mathbb{Q}}}) + \sum_{I_{\mathbb{Q}}, \mathbb{R}_{>0}w} \mathrm{mult}_{I_{\mathbb{Q}}, \mathbb{R}_{>0}w}(C.\mathcal{B}_{I_{\mathbb{Q}}, \mathbb{R}_{>0}w}). \tag{4.3.5}$$

We use $(C.\mathcal{Z}(m))$ to denote $\sum_{v \text{ non-Archimedean}} i_v(C.\mathcal{Z}(m)) \log |\mathbb{F}_v|$, the sum of local intersection numbers at all non-Archimedean places, with similar definitions for $(C.\mathcal{B}_{J_{\mathbb{Q}}})$ and $(C.\mathcal{B}_{I_{\mathbb{Q}}, \mathbb{R}_{>0}w})$. Note that $(C.\mathcal{B}_{J_{\mathbb{Q}}}), (C.\mathcal{B}_{I_{\mathbb{Q}}, \mathbb{R}_{>0}w})$ are both independent of m and thus the asymptotic estimates only involve $\mathrm{mult}_{J_{\mathbb{Q}}}(m), \mathrm{mult}_{I_{\mathbb{Q}}, \mathbb{R}_{>0}w}$. See §4.8.1 for details.

We remark that if one only needs to work with Hecke translates of a fixed finite set of $\mathcal{Z}(m_0)$'s (as in [Cha18, ST20][27]), one can use Hecke translates to pullback Green function on $\mathcal{Z}(m_0)$ to obtain Green functions on its Hecke translates pullback, which is also (a union of) special divisors. The asymptotic estimates, as the degree of the Hecke correspondence goes to ∞, behave in the same way as ϕ_m above as $m \to \infty$. Therefore by a slight abuse of notation, we still use ϕ_m to denote the Green function used in §4.4 obtained by pullback via Hecke correspondences.

4.4 Splitting of abelian surfaces over number fields: the case of real multiplication

In this section, we will describe the proof of Theorem 4.1. Let A/K denote an abelian surface over a number field admitting real multiplication by F. Without loss of generality, we assume that $\mathcal{O}_F \subset \mathrm{End}(A)$. Then A corresponds to a moduli point $x \in \mathcal{S}^{\Sigma}(\mathcal{O}_K)$, where $\mathcal{S}$ is the integral model of the Hilbert modular surface associated to F and $\mathcal{S}^{\Sigma}$ is a toroidal compactification. We will use the notation $x \in \mathcal{S}^{\Sigma}(\mathcal{O}_K)$ instead of $C \to \mathcal{S}^{\Sigma}$. We can and will assume that x_K does not lie on any $Z(m)$ (indeed, A mod v would be isogenous to a self-product of elliptic curves for all places of good reduction if $x_K \in \mathcal{Z}(m)$ — see [ST20, §5.2.2]).

While our proof in [ST20] is in terms of the Arakelov intersection of the Hecke orbit of x with a fixed finite sum of suitable special divisors, we will recast the proof in the context of considering the Arakelov intersection of x with a suitable sequence of special divisors.

4.4.1 Estimating the global intersection

Unlike in the higher dimensional (or characteristic p) case, we adapt Autissier's work [Aut05] and Borcherds theory to our setting to estimate the Arakelov intersection $(x, (\mathcal{Z}(m), \phi_m))$ (for appropriately chosen m). In particular, we prove that for appropriate choices of m, the Arakelov intersection $(x, (\mathcal{Z}(m), \phi_m)) \gg m \log m$. The precise statement is [ST20, Proposition 5.1.6].[28]

[27] From the statements of the theorems in [SSTT22, Tay24], it may appear to the reader that one may only work with a suitable infinite sequence of $\mathcal{Z}(m)$ (such as the Hecke translates of a fixed $\mathcal{Z}(m_0)$). However, our proof uses a so-called global height input to bound the local intersection multiplicities discussed in §4.6.2 and we need asymptotic estimate of $(C.(\mathcal{Z}(m)^{\Sigma}, \phi_m))$ for all $m \in \mathbb{Z}_{>0}$.

[28] We remark that the statement in *loc. cit.* is about the average Faltings height of a set of abelian surfaces isogenous to A. That result can be used in conjunction with Borcherds theory (summarized in the special case we need in [ST20, Lem. 5.1.1]) to deduce the estimate on $(x, \mathcal{Z}(m))$ provided here. More precisely, Borcherds theory relates the Faltings height of x to the Arakelov intersection of x with a fixed finite sum of special divisors. Therefore, the sum of the Faltings heights of x in a Hecke orbit of certain degree p is, up to a multiple of a fixed constant, the

To prove Theorem 4.1, we prove that given any finite set S of places v, there exists an infinite sequence of (suitably chosen) positive integers m such that $i_v(x.\mathcal{Z}(m)) = o(m \log m)$ for all non-Archimedean places $v \in S$, and such that $\phi_m(x^\sigma) = o(m \log m)$ for all Archimedean places σ of K.

We use a suitable sequence of integers m_i such that $Z(m_i)$ is a compact divisor in $\mathcal{S}_{\mathbb{Q}}$.[29] This allows us to ignore the boundary of $\mathcal{S}$ in $\mathcal{S}^\Sigma$ as all the intersections happen in $\mathcal{S}$. In fact, we choose $m_i = rp_i$, where r is from a fixed finite set of positive integers (as the set is finite and we will focus on asymptotics, so by abuse of notation, we dropped the subscript i in r), and the p_i are primes which split as a product of principal prime ideals in $\mathcal{O}_F$.

In the next two subsubsections, we describe how we establish the bounds for Archimedean and non-Archimedean places respectively.

4.4.2 Bounding the contribution from Archimedean places.

The strategy to bound the Archimedean contribution is as follows. We will work with a suitably large auxiliary integer N to keep track of the sizes of quantities involved.

Step 1 Let m be a suitably chosen integer (see §4.4.1 for the description) in the range $[N^{1/2}, N]$ such that $\phi_m(x^\sigma) \asymp m \log m$,[30] then x^σ is very close to a formal analytic component of $Z(m)$ relative to the hyperbolic metric on $\mathcal{S}(\mathbb{C})$ (see [ST20, Lemma 3.1.3] for the precise definition).[31] Here, $x^\sigma \in \mathcal{S}(\mathbb{C})$ is the $\mathbb{C}$-point induced by the Archimedean place σ. This reduction step is essentially the proof of [ST20, Thm. 3.2.1] using the equidistribution of Hecke orbits in $\mathcal{S}(\mathbb{C})$ (see [COU01]).

Step 2 Let m_1 and m_2 be two suitably chosen integers in the range $[N^{1/2}, N]$ such that $\phi_{m_i}(x^\sigma) \asymp m_i \log m_i$. Then there exists a special point (i.e., CM point) y on $\mathcal{S}(\mathbb{C})$ which satisfies the following properties:

- The point y is close to x^σ relative to the hyperbolic metric on $\mathcal{S}(\mathbb{C})$ ([ST20, Lemma 3.1.5]).
- The lattice of special endomorphisms of y has rank 2 and thus gives rise to a positive definite binary $\mathbb{Z}$-coefficient quadratic form Q_y. This quadratic form represents m_1, m_2 (see *loc. cit.*).

Arakelov intersection of x with the Hecke translate of the same degree of this fixed finite sum of special divisors. Since the Hecke translate of special divisors is a finite union of special divisors $\mathcal{Z}(m)$ with $m \asymp p$, there must be at least one such m such that $(x, (\mathcal{Z}(m), \phi_m)) \gg m \log m$. We would like to mention that the idea of relating Faltings height with arithmetic intersection with special divisors using Borcherds theory has already appeared in Yang's work on special case of Clomez conjecture for abelian surfaces [Yan10].

[29] There are many compact special divisors in Hilbert modular surfaces (see for instance [ST20, Prop. 2.1.2, Cor. 2.1.3]). The inputs from Borcherds theory are robust enough that we may work exclusively with compact divisors (see [ST20, Lemma 5.1.1] for the precise statement).

[30] The asymptotic is understood as $N \to \infty$ — for instance, see [ST20, Thm. 3.2.1] for a precise formulation. Note that *loc. cit.* is formulated in terms of the sum of the Green function for a fixed finite sum of special divisors evaluated on points in a Hecke orbit. However, the fact that we define the Green function for $\mathcal{Z}(m)$ in terms of the pullback under a Hecke correspondence implies that this sum is exactly the value of the Green function of $\mathcal{Z}(m)$ at x.

[31] More precisely, as recalled in [ST20, §2.1.1], the preimage of $Z(m)$ in $\mathbb{H}^2$ (here $\mathbb{H}$ denotes the upper half plane) is a union of geodesics and we refer to these geodesics as "formal analytic components" of $Z(m)$. [ST20, Lemma 3.1.3] shows that having a point in the Hecke orbit of x^σ of degree p being close to $Z(m_0)$ is equivalent to x^σ close to a formal analytic component of $Z(pm_0)$ and thus we can translate the statements on points in the Hecke orbit as in [ST20] to the results on x as stated in this subsubsection.

- The discriminant of Q_y is $O(N^2)$ ([ST20, Proof of Lemma 3.1.6]).

Step 3 Let m'_1, m'_2 be another pair of positive integers satisfying the hypotheses of Step 1. Then the special point y' constructed as above is the same as the point y ([ST20, Lemma 3.1.6]). Consequently Q_y represents m'_1 and m'_2.

Step 4 The set of integers represented by a positive definite *binary* quadratic form has density zero. As the set of integers m_i that we work with is also density zero, we cannot directly use this fact to establish our bounds. Instead, we use a quantitative result to show that among integers $m \in [N^{1/2}, N]$ that we consider, the proportion of those m which are not represented by Q_y is approaching 1 as $N \to \infty$ since $\operatorname{disc} Q_{y_N} \to \infty$, where y_N is the (unique) CM point associated to N. ([ST20, Claim 3.1.9]).

The strategy outlined above yields that as the auxiliary parameter N grows to infinity, our initial point x satisfies $\phi_m(x^\sigma)$ is $o(m \log m)$ for most suitable integers $m \in [N^{1/2}, N]$. The precise statement is [ST20, Theorem 3.2.1].

4.4.3 Bounding the non-Archimedean contribution

Our bounds for $i_v(x, \mathcal{Z}(m))$, where v is a p-adic place, are purely local, i.e., we do not need to assume that x is defined over a number field. For ease of exposition, we assume that $x \in \mathcal{S}(W)$, where W is the ring of integers of an unramified extension of $\mathbb{Q}_p$.[32] Let A/W denote the abelian surface induced by the point x. We introduce the following notation:

1. Let A_n denote the abelian surface mod p^n, and let $\mathscr{G}_n = A_n[p^\infty]$.
2. Let L_n denote the lattice of special endomorphisms of A_n, and let Λ_n denote the $\mathbb{Z}_p$-lattice of special endomorphisms of $\mathscr{G}_n$. We note that from Definition 3.3, an endomorphism of $\mathscr{G}_n$ is a special endomorphism if and only if it is an element of $L_1 \otimes \mathbb{Z}_p$. The Serre–Tate lifting theorem yields that $L_n = \Lambda_n \cap L_1$, where the intersection occurs in Λ_1.
3. Let $\mathscr{G} = A[p^\infty]$, and let Λ denote the $\mathbb{Z}_p$-module of special endomorphisms of $\mathscr{G}$. We note that $\Lambda = \cap_n \Lambda_n$, and this intersection can be non-zero even though we have $\cap_n L_n = \{0\}$ from our assumption on $x_K \notin \cup_{m \in \mathbb{Z}_{>0}} Z(m)$.
4. Let $\mu_1(n) \le \mu_2(n) \le \cdots \le \mu_{\operatorname{rk} L_n}(n)$ denote the successive minima of the quadratic lattice L_n.

We need to prove that $i_v(x, \mathcal{Z}(m_i)) = o(m_i \log m_i)$ for a suitable sequence of positive integers m_i. That corresponds to establishing the following bound:

$$\sum_{n=1}^{\infty} \#\{\lambda \in L_n : Q(\lambda) = m_i\} = o(m_i \log m_i).$$

The main term of the LHS is entirely controlled by the ranks and discriminants of the lattices involved. Indeed, we prove the following result:

[32] The proof for the general case (namely with ramified extensions) is essentially the same as far as one keeps track of the ramification index and we refer the reader to [ST20, §4] for details.

Theorem 4.9. *The ranks of Λ and L_1 are bounded above by 2 and 4 respectively, with equality holding only if A_1 is supersingular. Furthermore, there exists some positive integer n_0 such that $\Lambda_{n+n_0} = \Lambda + p^n \Lambda_{n_0}$ for all $n \geq 0$. In particular, all L_n's have the same rank.*

Proof. The first claim is [ST20, Lemma 4.3.2] and the second claim is [ST20, Theorem 4.1.1] proved using Grothendieck–Messing theory. The last claim follows from the second claim and that $L_n = \Lambda_n \cap L_1$. □

Theorem 4.9 is not sufficient to establish the required bounds, because it is *a priori* possible that Λ is very well approximated by sublattices M'_n of L_1 having small discriminant. Indeed, there could exist an integer n, and a sublattice $L' \subset L_1$, such that $L' \equiv \Lambda \mod p^N$ with N arbitrarily large relative to n — if this were to happen, we would have that $L' \subset L_{n'}$ for every $n' \leq N$. This would imply that if $\lambda \in L'$ satisfied $Q(\lambda) = m$, then $i_v(x, \mathcal{Z}(m)) \gg N$. Equivalently, while the index of L_n in L_1 grows rapidly with n (order of growth p^{2n}), it is still possible that some of the successive minima (in this case, $\mu_1(n), \mu_2(n)$) grow extremely slowly. Note that the fact that Λ has rank at most 2 implies that $\mu_3(n), \mu_4(n)$ grow exponentially in n. We are going to prove that most of m are not representable by L' as above. To achieve this, let $\lambda_1(n), \lambda_2(n)$ denote the two linearly independent vectors of L_n with lengths $\mu_1(n), \mu_2(n)$ respectively.[33]

Fix some auxiliary integer N, which will eventually grow to infinity. We now outline our strategy:

Step 1 Consider the set of suitable integers $m \in [N^{1/2}, N]$ (here suitable means in the sense of §4.4.1) represented by L_n for n, where we set n to approximately equal $C \log N$, where C is a large absolute constant. Since $\mu_3(n), \mu_4(n)$ grow exponentially in n, then m must be represented by L'_n, which is the lattice spanned by $\lambda_1(n), \lambda_2(n)$.

Step 2 Recall that $\cap_n L_n = 0$. Therefore $\mu_1(n)$ tends to infinity, as does the discriminant of L'_n. Since the rank of Λ, and therefore that of L'_n, is at most 2. By the same argument as in Step 3 and Step 4 in §4.4.2, the proportion of suitable $m \in [N^{1/2}, N]$ not represented by L'_n grows to 1. (We remark that here we make crucial use of the rank 2 fact, which fails in higher dimensional case. In other words, we cannot use this method to rule out a density approaching 0 set of m satisfying the hypothesis in Step 1. See §4.6.1 for details.)

Step 3 To conclude, by working with the most suitable $m \in [N^{1/2}, N]$ given in Step 2, we conclude that m is not representable by L_n with $n \geq C \log N$ (here C is the constant in Step 1) and hence

$$i_v(x, \mathcal{Z}(m)) = \sum_{n=1}^{C \log N} \#\{\lambda \in L_n : Q(\lambda) = m_i\}.$$

This strategy is carried out in [ST20, Theorem 4.3.4] and philosophically speaking, this result bounds the error term of $i_v(x, \mathcal{Z}(m))$ for most suitable m.

The main term in the local intersection multiplicity $\sum_{n=1}^{C \log N} \#\{\lambda \in L_n : Q(\lambda) = m_i\}$ is controlled by Theorem 4.9 via a direct application of geometry-of-numbers argument. This is carried out in [ST20, Theorem 4.3.3] (see also §4.6.2 Step 3 for a similar argument).

[33] Note that the case when $\mathrm{rk}\, L_n = 4$ is the key case in our proof and the reader shall feel free to stick to this case. We adopt the convention that terms with $\mu_i(n)$ with $i > \mathrm{rk}\, L_n$ are ignored.

Remark 4.10. We remark that Λ having rank 2 is equivalent to the moduli point x being the p-adic limit of a sequence of CM points (indeed, with each CM point being isogenous to a self-product of elliptic curves). In the case of $\mathcal{A}_2$, i.e., the question of splitting of abelian surfaces with trivial endomorphism ring, the point x being the p-adic limit of such CM points would imply that the $\mathbb{Z}_p$-lattice of special endomorphisms of the p-divisible group associated to x would have rank 3.

Remark 4.11. We remark that similar results hold for all GSpin Shimura varieties associated to integral quadratic forms with signature $(2, 2)$. Roughly speaking, there are three cases. The first is when the quadratic form is split over $\mathbb{Q}$. The Shimura variety is a product of modular curves, and this is just Charles' result. The second is when the maximal isotropic $\mathbb{Q}$-subspace of the quadratic form is one-dimensional. This corresponds to the case of Hilbert modular surfaces — namely the case just discussed in this section. Finally, the last remaining case is when the quadratic form is anisotropic. The Shimura variety is compact in this setting. The methods of [ST20] completely go through in this case — indeed, the use of Borcherds theory to find compact special divisors is unnecessary as every special divisor is automatically compact.

4.5 Splitting of abelian surfaces over function fields

In this section, we will describe the proof of Theorem 4.2. We will focus on the case when $A/\overline{k(C)}$ has no extra endomorphisms. It suffices to prove that the image of C intersects $\bigcup_{m\in\mathbb{Z}_{>0}} \mathcal{Z}(m^2)$ at infinitely many points, and the strategy will be the same local-global strategy. We set up the following notation. Let $P \in C$ denote a closed point, and let $t \in \mathcal{O}_C$ denote some uniformizing parameter at P. We will also use $A/\overline{\mathbb{F}}_p[[t]]$ to denote the abelian surface restricted to the formal neighbourhood of C at P. Analogous to Section 4.4.3, let:

- A_n denote $A \bmod t^n$ and $\mathscr{G}_n = A_n[p^\infty]$,
- L_n be the lattice of special endomorphisms of A_n, and Λ_n denote the $\mathbb{Z}_p$-lattice of special endomorphisms of $\mathscr{G}_n$.

4.5.1 Differences with the number field case.

There several differences (some of which add significant difficulties) when compared to the proof of Theorem 4.1.

1. The intersection theoretic framework is classical and the residue characteristic of the closed points do not grow. Consequently, the global intersection number $(C.\mathcal{Z}(m^2))$ grows as m^3 and not $m^3 \log m$, as $m \to \infty$.
2. The local intersection numbers at supersingular points P have order of growth $i_P(C.\mathcal{Z}(m^2)) \asymp m^3$. Therefore unlike in the number field case, we have to keep track of coefficients of leading order terms for the local and global intersection numbers, and prove that the sum of local coefficients is strictly smaller than the global coefficient. In particular, a characteristic p analogue of Theorem 4.9 with an unspecified n_0 is not sufficient to bound $i_P(C.\mathcal{Z}(m^2))$ for supersingular P.
3. The equicharacteristic p deformation theory of special endomorphisms (and endomorphisms in general) is more difficult than its mixed characteristic analogue, because Grothendieck–Messing theory no longer applies. This is because the ideal $(t) \subset \overline{\mathbb{F}}_p[[t]]$ does not admit

divided powers, and therefore Grothendieck–Messing theory does not apply. Moreover, the analogue of Theorem 4.9 (with t in place of p, even with an unspecified n_0) is not even true! (See [MST22b, §5.3] for details.) This compounds the second difficulty, because we now have to more precisely compute a quantity in a more difficult setting.

4. As the abelian surface is not isotrivial and generically ordinary CM abelian varieties in characteristic p are all defined over finite fields, $A/\overline{\mathbb{F}}_p[[t]]$ is *not* the t-adic limit of CM abelian schemes over $\overline{\mathbb{F}}_p[[t]]$, unlike the setting of Remark 4.10. Consequently, the rank of Λ is at most 2.[34] It is precisely this feature that enabled us to prove Theorem 4.2 in the setting of $\mathcal{A}_2$ using purely local methods to estimate the local intersection numbers $i_P(C, \mathcal{Z}(m^2))$ — i.e., we do not need to use the fact that $A/\overline{\mathbb{F}}_p[[t]]$ descends to the global field $k(C)$.

5. The fact that we sum over squares complicates the geometry-of-numbers arguments needed.

4.5.2 Bounding local intersection multiplicities

We now detail how we overcome some of these added difficulties. As the hardest case is of supersingular points, we henceforth assume that the image of P in $\mathcal{A}_2$ is supersingular. In order to compute precise constants for the global intersection number, we use arithmetic Borcherds theory following Howard–Madapusi [HP20] (recalled in §3.7), work of Bruinier–Kuss [BK01], Siegel mass formula, and local density estimates using work of Hanke [Han04], to explicitly compare the Fourier coefficients of the generating series $-(C.\omega) + \sum_{m\in\mathbb{Z}_{>0}}(C.\mathcal{Z}(m))q^m$ with that of the theta series associated to the lattice of special endomorphisms associated to a superspecial[35] abelian surface. This is done in [MST22, Section 4], and the precise comparison result is a combination of §§4.1.4, 4.2.1, Lemmas 4.3.2 and 4.4.6 of *loc. cit.*[36]

In order to estimate the main term of $i_P(C, \mathcal{Z}(m^2))$, we must precisely estimate the discriminant of L_n in terms of n and the starting data of $A/\overline{\mathbb{F}}_p[[t]]$. We do this by using work of Kisin [Kis10, Sections 1.4-1.5] to write out the F-crystal $\mathbf{L}_{\mathrm{cris}}$ (see §3.2 for definition) over $\overline{\mathbb{F}}_p[[t]]$. This is a module $\mathbb{L}$ with semilinear Frobenius over the ring $W(\overline{\mathbb{F}}_p)[[t]]$. We then use the group-theoretic description of Frobenius to explicitly calculate the Frobenius invariant sections in $\mathbb{L} \otimes W(\overline{\mathbb{F}}_p)[1/p][[t]]$. This gives us precise control over the discriminant of L_n. This is the technical heart of the paper, and is done in [MST22, Sections 5 and 6] and the appendix of [MST20, Appendix A], and the precise statements are [MST22, Definition 5.1.1, Theorem 5.1.2, and Proposition 5.1.3] and [MST20, Theorem A.0.1].

A crucial aspect of our calculation is that our estimate of the ratio $\mathrm{Disc}(L_n)/\mathrm{Disc}(L_1)$ is in terms of[37] the vanishing of the Hasse invariant restricted to $\mathrm{Spec}\,\overline{\mathbb{F}}_p[[t]]$. This dependence is precisely what allows us to compare the coefficients of leading order terms of $i_P(C.\mathcal{Z}(m))$ and $(C.\mathcal{Z}(m))$. Indeed, upon comparing with the m-th Fourier coefficient $q(m)$ of the Eisenstein series in Bruinier–Kuss (see for instance [MST22, §4.1.4] for the definition of E_0), the leading order term of $(C.\mathcal{Z}(m))$ is seen to equal $(C.\omega)q(m)$. Recall from §3.7 that ω is the Hodge bundle. On the other hand, the divisor class H of the Hasse invariant is $(p-1)\omega$. Since the vanishing locus of the Hasse invariant is the non-ordinary locus, the total number of non-ordinary points with multiplicities (the multiplicities

[34] The rank of Λ being equal to 2 would imply that the $\overline{\mathbb{F}}_p[[t]]$-valued moduli point associated to A is the t-adic limit of $\overline{\mathbb{F}}_p[[t]]$-points contained in special divisors $\mathcal{Z}(m)$ with varying m.

[35] An abelian variety is said to be superspecial if it is isomorphic to a self-product of supersingular elliptic curves.

[36] The sizes of the Fourier coefficients are asymptotic to the Fourier coefficients of the corresponding Eisenstein series; Lemma 4.4.6 in *loc. cit.* gives the comparison of the Fourier coefficients of the corresponding Eisenstein series.

[37] Of course, the ratio also depends on n.

are given by the order of vanishing of the Hasse invariant) is $(p-1)(C.\omega)$. Therefore, we have the following estimate:

$$(C.\mathcal{Z}(m)) \sim \frac{q(m)}{p-1}(C.H) = \sum_{P\in C} \frac{q(m)}{p-1} i_P(C.H) \tag{4.5.1}$$

Note that the sum above is over non-ordinary points of C, which also include supersingular points. Therefore, (4.5.1) allows us to compare the coefficients of leading order terms of $\sum_{P\in C} i_P(C.\mathcal{Z}(m))$ and $(C.\mathcal{Z}(m))$, by comparing the coefficients of leading order terms of $i_P(C.\mathcal{Z}(m))$ and $\frac{q(m)}{p-1} i_P(C.H)$ at a fixed supersingular point $P \in C$.

A feature crucial to this comparison is the dichotomy between superspecial and supergeneric points (here we refer to nonsuperspecial supersingular points as supergeneric). The quantity $\mathrm{Disc}(L_1)$ is smaller at superspecial points than at supergeneric points and so $i_P(C.\mathcal{Z}(m))$ is larger at superspecial points than at supergeneric points. Since $\mathrm{Disc}(L_1)$ is large enough at supergeneric points, our estimate of the discriminant of L_n mentioned above is sufficient to show that the main term in $i_P(C.\mathcal{Z}(m))$ is smaller than $\frac{q(m)}{(p-1)} i_P(C.H)$.

We notice that the vanishing locus of the Hasse invariant (i.e. the non-ordinary locus) is singular at superspecial points (and this is also reflected in our local calculation referred as the existing of a special endomorphism with very rapid decay — see [MST22, Definition 5.1.1, Theorem 5.1.2]). It is this that yields the fact that $\frac{q(m)}{p-1} i_P(C.H)$ is larger than the main term in $i_P(C.\mathcal{Z}(m))$ at a superspecial point P. More precisely, the vanishing order of the Hasse invariant is at least twice[38] as large as the smallest n for which $L_n \neq L_1$; the contribution from all n with $L_n = L_1$ is a good approximation of $i_P(C.\mathcal{Z}(m))$ for $p \gg 1$. To summarize, in order to show that the main term in $i_P(C.\mathcal{Z}(m))$ is smaller than the proportion in the global intersection number associated to a supersingular point on C, we use the distinct features of superspecial and supergeneric points, namely the first is a singular point in the non-ordinary locus and the second has larger $\mathrm{Disc}(L_1)$.

The proof of [MST22b, Prop. 7.17][39] shows how we use these two features to prove that the sum of the main terms in $i_P(C.\mathcal{Z}(m))$ over all supersingular points P on C is at most $\alpha(C.\mathcal{Z}(m))$ for some absolute constant $0 < \alpha < 1$ as $m \to \infty$. See also [MST22b, §3.4] for a heuristic argument highlighting these two features (note that here we do not distinguish whether we work with all m or only squares as the method and heuristics in the computation of the main term is the same in both cases).

In order to deal with the error terms of $i_P(C.\mathcal{Z}(m^2))$, we use a carefully refined version of the strategy carried out in [ST20, Theorem 4.3.4] to account for the fact that we are summing over squares. Of course, our argument only works because Λ has rank at most 2 (see Remark 4.10). See [MST22, Prop. 9.1.3] for the input to treat squares and §9.2 in *loc. cit.* for the proof — see also Remark 4.13 (3) for the number field case.

To summarize, the above discussions indicate how we prove that the sum of $i_P(C.\mathcal{Z}(m))$ over all supersingular points P on C is at most $\alpha(C.\mathcal{Z}(m))$ for some absolute constant $0 < \alpha < 1$. Note that (2) in Section 4.5.1 is only for supersingular points. Indeed, we prove that on average over m, $i_P(C.\mathcal{Z}(m)) = o((C.\mathcal{Z}(m)))$ (see [MST22, §9.3]) and thus we conclude our argument as in §4.4 to obtain Theorem 4.2.

[38]This is true as $p \to \infty$. See the definition of very rapid decay for the precise statement.

[39]Here we give a reference to our later paper as this part of the computation in [MST22] is mixed with other estimates in [MST22, §9.2].

4.6 The general case

In this subsection, we describe the proofs of Theorems 4.3 and 4.4, mainly focusing on the extra difficulties and the new ideas required to overcome them. Recall that $\mathcal{S}$ is the integral model of the GSpin Shimura variety $\mathcal{S}_{\mathbb{Q}}$ associated to an integral quadratic form with signature $(b,2)$ (see §3.1).

4.6.1 Transcendental examples

We will first briefly describe an example of a (transcendental) point $y \in \mathcal{S}(W(\mathbb{F}_q))$, where $q = p^r$, that has large local intersection multiplicities with $\mathcal{Z}(m)$ for many values of m. Here $\dim \mathcal{S}_{\mathbb{Q}} = b$ is sufficiently large. More specifically, we will construct y such that there exists an infinite sequence $\{m_i\}_{i=1}^{\infty}$ that has the property $i_p(y,\mathcal{Z}(m)) \gg m^{\frac{b}{2}} \log m$ (we will see in §4.6.2 that $m^{\frac{b}{2}}\log m$ is the asymptotic of the global intersection number defined in §4.3) for a density-one set of integers m in the interval $[m_i, m_i^2]$. Indeed the example also has the property that the function $d(M) = \frac{\#\{m<M : m^{\frac{b}{2}} \log m = o(i_p(y.\mathcal{Z}(m)))\}}{M}$ satisfies $\limsup_{M\to\infty} d(M) = 1$. Let Λ be the $\mathbb{Z}_p$-lattice of special endomorphisms of the p-divisible group associated to y and let L_1 be the $\mathbb{Z}$-lattice of special endomorphisms of the abelian variety associated to $y_{\mathbb{F}_q}$ as in Section 4.4.3. It is possible to choose the point y (for example, using Serre–Tate coordinates) such that Λ has large rank and is p-adically well approximated by $\mathbb{Z}$-sublattices of L_1. More specifically, it is possible to find a point y and a sequence of saturated sublattices $L_i' \subset L_1$ having discriminants D_i such that $L_i' \equiv \Lambda \bmod p^{N_i}$, where N_i is (for example) asymptotic to $e^{e^{D_i}}$. Let $\lambda_i \in L_i'$ with $Q(\lambda_i) = m_i$. Then $i_p(y,\mathcal{Z}(m_i)) \geq N_i$. Assuming that Λ has large enough rank, it is easy to arrange for L_i' to represent a density-one set of integers having size around e^{D_i}. Therefore, if m_i is one of these integers, we see that $i_p(y,\mathcal{Z}(m_i))$ is at least $\asymp e^{m_i}$. Therefore we have $i_p(y,\mathcal{Z}(m_i)) \gg m_i^{\frac{b}{2}} \log m_i$. Here is what happens in terms of the lattices L_{N_i}. The analogue of Theorem 4.9 is still true (see Theorem 4.12). This theorem gives extremely good control on the discriminants of L_{N_i}. However, the discriminant of a lattice only gives a first-order approximation of the number of lattice points with bounded norm. In order to get good bounds, one needs to effectively lowerbound the successive minima of L_{N_i}. In the example outlined above, L_{N_i} contains lower-rank sublattices (namely, L_i') having small discriminants. Therefore, the first several successive minima of L_{N_i} can be extremely small, even though the discriminant itself is very large. This construction is carried out in detail in [SSTT22, Section 7.3].[40] For the characteristic p case, see [MST22b, Section 3.5]. Similarly, there exists points $y \in \mathcal{S}(\mathbb{C})$ such that the Archimedean intersection of y with $\mathcal{Z}(m_i)$ is large relative to m_i.[41]

4.6.2 The global input

Our exposition will mainly focus on the number field case with $b \geq 3$ (for $b = 2$, see Remark 4.11). As in Section 4.4, we will use the notation $x \in \mathcal{S}(\mathcal{O}_{K,S})$. A place v will denote a place of K not in S with residue characteristic p, and we postpone the case of when $v \in S$ to Section 4.8. We also assume that $x_K \notin \cup_{m\in\mathbb{Z}_{>0}} \mathcal{Z}(m)$. Otherwise, since every $\mathcal{Z}(m)$ is a GSpin Shimura variety, we may work with the smallest $\mathcal{S}$ that has the property that x_K does not lie on any special divisor.

For the ease of notation, we assume that $x \in \mathcal{S}(\mathcal{O}_K)$ and therefore consider the arithmetic intersection of x with $\widehat{\mathcal{Z}}(m)$ in $\mathcal{S}$. In the general case, we work in a suitable $\mathcal{S}^{\Sigma}$, after an analysis of

[40] The construction there also indicates that $\mathrm{rk}\,\Lambda \geq 5$ is what we mean by rank large enough.

[41] Note that as indicated in Remark 4.13 (2), the main term in ϕ_m behaves similarly to the local intersection number at non-Archimedean places.

boundary components in $\mathcal{Z}^\Sigma(m)$. The discussion on local intersection multiplicities in the setting of $x \in \mathcal{S}(\mathcal{O}_K)$ will still hold in the general case for primes of good reduction and Archimedean places (see §4.8.1).

Recall from §§3.7,4.3, $\widehat{\mathcal{Z}}(m) = (\mathcal{Z}(m), \Phi_m)$ and $\Phi_m = \phi_m - C_m$. By the modularity of generating series of $\widehat{\mathcal{Z}}(m)$ recalled in §3.7, we have $(x.\widehat{\mathcal{Z}}(m)) = O(m^{b/2})$. Using explicit formula for C_m in [BK03], [SSTT22, Prop. 5.2] shows that $C_m \asymp m^{b/2} \log m$. Therefore the global intersection number

$$(x.(\mathcal{Z}(m), \phi_m)) \asymp m^{b/2} \log m. \tag{4.6.1}$$

The first input to bound $i_v(x.\mathcal{Z}(m))$ is the following analogue of Theorem 4.9. Notation as in §4.4.3 and we also work with the unramified case for the ease of exposition. Let L_n denote the lattice of special endomorphisms of $x \bmod p^n$, and let Λ_n denote the $\mathbb{Z}_p$-lattice of special endomorphisms of the p-divisible group associated to $x \bmod p^n$. We have $L_n = \Lambda_n \cap L_1$, where the intersection occurs in Λ_1. Let Λ denote the $\mathbb{Z}_p$-module of special endomorphisms of the p-divisible group associated to $x_{W(\overline{\mathbb{F}}_p)}$.

Theorem 4.12. *The ranks of Λ and L_1 are bounded above by b and $b+2$ respectively, with equality holding only if $x \bmod p$ is supersingular. Furthermore, there exists some positive integer n_0 such that $\Lambda_{n+n_0} = \Lambda + p^n \Lambda_{n_0}$ for all $n \geq 0$. In particular, all L_n's have the same rank.*

The example in Section 4.6.1 unequivocally demonstrates that in the general case, it is not possible to bound local intersection multiplicities (even on average) using purely local arguments. We expect that $i_v(x, \mathcal{Z}(m_i)) = o((x.\mathcal{Z}(m_i)))$, for any increasing sequence of integers m_i which satisfy $\mathcal{Z}(m_i) \neq \emptyset$, where $x \in \mathcal{S}(K)$, K is a number field, and v is a non-Archimedean place of K.[42] This turns out to be an extremely interesting question in transcendence which we will discuss in Section 5.1. However, showing that the example of $y \in \mathcal{S}(W(\mathbb{F}_q))$ in 4.6.1 cannot be defined over a number field is easier than proving that $i_v(x, \mathcal{Z}(m_i)) = o((x.\mathcal{Z}(m_i)))$. Indeed, by [SSTT22, Theorem 5.8 (ii)], the local intersection multiplicity

$$i_v(x.\mathcal{Z}(m)) \ll m^{b/2} \log m. \tag{4.6.2}$$

On the other hand, $i_v(y.\mathcal{Z}(m))$ is exponential in m for an infinite set of integers m (indeed, for a set of integers having upper density 1). It follows that y must be a transcendental point.

The intuition behind (4.6.2) is that the local intersection multiplicity must be bounded by the global intersection number. Indeed, by Equations (4.3.2) and (4.6.1), and that $i_v(x.\mathcal{Z}(m)) \geq 0$ for all non-Archimedean v, the bound (4.6.2) follows from a lower bound on $\phi_m(x^\sigma)$ for all Archimedean places σ. A good enough such bound follows from [SSTT22, Prop. 5.4]. We also remark that as $i_v(C.\mathcal{Z}(m)) \geq 0$ holds for all places in the case of function fields, we directly obtain $i_v(C.\mathcal{Z}(m)) \ll m^{b/2}$. (Note that since the Hodge line bundle ω is ample on $\mathcal{S}_{\mathbb{F}_p}$ and its minimal/Baily–Borel compactification, $(C.\omega) > 0$ and then by modularity results in §3.7, we have $(C.\mathcal{Z}(m)) \asymp m^{b/2}$.)

As remarked in §4.6.1, Theorem 4.12 gives a good control on $\operatorname{Disc} L_n$ and hence gives the first-order approximation of $i_v(x.\mathcal{Z}(m)) = \sum_{n=1}^{\infty} \#\{\lambda \in L_n : Q(\lambda) = m\}$. We now describe how the bound Equation (4.6.2) is leveraged to bound the error terms of $i_v(x.\mathcal{Z}(m))$ in the proof of Theorem 4.3. We remark that if we only want to prove that x and $\cup_{m \in \mathbb{Z}_{>0}} \mathcal{Z}(m)$ intersect at infinitely many

[42] As we see from Section 4.5, this is not true over function fields when v corresponds to certain supersingular points.

primes (not the stronger statement with $\cup_{m\in\mathbb{Z}_{>0}}\mathcal{Z}(m^2)$), the following argument is sufficient to control local intersection at v and we do not need Theorem 4.12.

The key idea is to use Equation (4.6.2) for every m to obtain strictly better bounds on average, i.e., to prove that

$$\sum_{m<M} i_v(x.\mathcal{Z}(m)) = o(\sum_{m<M} m^{b/2}\log m). \tag{4.6.3}$$

Recall the setup in Section 4.4.3. Let $\mu_i(n)$ denote the ith successive minima of L_n. We assume that x mod v is supersingular (this is the key case and all other cases follow from the same argument). Then $\dim L_n = b+2$. Recall that the local intersection number is $i_v(x.\mathcal{Z}(m)) = \sum_{n=1}^{\infty}\#\{\lambda\in L_n : Q(\lambda)=m\}$ as $x\notin\mathcal{Z}(m)$. We implement our idea in the following steps:

Step 1 Use Equation (4.6.2) to obtain lower bounds on $\mu_i(n)$. Let $\lambda_n\in L_n$ denote a vector such that $\mu_1(n)^2 = Q(\lambda_n) = m_0$. Since $\lambda_n\in L_n\subset L_{n'}$ for $n'<n$, we have that $i_v(x.\mathcal{Z}(m_0))\geq n$, and thus $m_0^{b/2}\log m_0 \gg n$. Therefore, we have that $(\mu_1(n)^2)^{b/2}\log(\mu_1(n)^2)\gg n$ and thus $\mu_1(n)\gg n^{1/(b+\epsilon)}$ (the ϵ is to account for the log term). Then by definition

$$\mu_i(n)\gg\mu_1(n)\gg n^{1/(b+\epsilon)}. \tag{4.6.4}$$

This is the content of [SSTT22, Lemma 7.6].

Step 2 We use Equation (4.6.4) to truncate the sum $\sum_{n=1}^{\infty}\#\{\lambda\in L_n : Q(\lambda)=m\}$ effectively in terms of m. Indeed, $\{\lambda\in L_n : Q(\lambda)=m\}\neq\emptyset$ implies $\mu_1(n)^2\leq m$ and thus $n\ll m^{\frac{b+\epsilon}{2}}$. In other words, there exists a large constant C_1 (only depending on x and v) such that $i_v(x.\mathcal{Z}(m)) = \sum_{n=1}^{N}\#\{\lambda\in L_n : Q(\lambda)=m\}$, with $N=[C_1 m^{\frac{b+\epsilon}{2}}]$.

Step 3 The bounds on average are deduced from the following calculation. For ease of notation, let $a_i(n)=\prod_{j=1}^{i}\mu_j(n)$. Then by [EK95], we have

$$\sum_{n=1}^{[C_1 M^{\frac{b+\epsilon}{2}}]}\#\{\lambda\in L_n : Q(\lambda)\leq M\}\ll\sum_{i=1}^{b+2}\sum_{n=1}^{[C_1 M^{\frac{b+\epsilon}{2}}]}\frac{M^{i/2}}{a_i(n)}. \tag{4.6.5}$$

Substituting the bounds $a_i(n)\gg n^{\frac{i}{b+\epsilon}}$ from Equation (4.6.4), we obtain that the inner sum for $i\leq b+1$ is bounded above by $O(M^{\frac{b+1}{2}})$, and for $i=b+2$, it is bounded by $O(M^{\frac{b+2}{2}})$. On the other hand, $\sum_{m=1}^{M} m^{b/2}\log m\gg M^{\frac{b+2}{2}}\log M$! Therefore, the individual bounds of $m^{b/2}\log m$ on $i_v(x.\mathcal{Z}(m))$ translate into a strictly better bound on average!

The above discussion provides the input to the proofs of Theorems 4.3 and 4.4. The following remark discusses other inputs to finish the proofs.

Remark 4.13. 1. There is a geometric reason that buttresses the fact that the global bounds on individual m give strictly better bounds on average. Loosely speaking, suppose there were "several" integers m_i, having roughly the same size, such that x was v-adically close enough to $\mathcal{Z}(m_i)$ to make the global bounds on $i_v(x.\mathcal{Z}(m_i))$ in Equation (4.6.2) sharp. Then x must also be v-adically close to $\cap\mathcal{Z}(m_i)$ — a special subvariety having high codimension. However, given m_i having roughly the same size, one would expect the high codimensional special subvariety $\cap\mathcal{Z}(m_i)$ also be contained in $\mathcal{Z}(m)$, where now $m=o(m_i)$ — this would result in x being

v-adically close to $\mathcal{Z}(m)$ in a way that would violate the global bound, as $m = o(m_i)$. Of course, this does not translate directly into a proof, but it is precisely this geometry that underlies the calculation outlined above.

2. For Archimedean places, something similar happens. In order to overcome the transcendental obstructions at Archimedean places σ, we again use the existence of global bounds for every $\phi_m(x^\sigma)$ to obtain strictly better bounds on average. We remark that the Archimedean setting shares intrinsic features with the non-Archimedean setting discussed above. The key quantity to be estimated is ([SSTT22, Prop. 5.4, (5.7)])

$$A(m,x) = 2 \sum_{\substack{\lambda \in L \\ Q(\lambda)=m \\ |Q(\lambda_x)| \le m}} \log\Big(\frac{m}{|Q(\lambda_x)|}\Big),$$

where L is the quadratic lattice used to define $\mathcal{S}$ in §3.1, the preimage[43] of x^σ in D_V ($V = L \otimes \mathbb{Q}$) determines a negative definite plane in $L \otimes \mathbb{R}$ and λ_x denotes the projection of λ to this plane. A $\lambda \in L$ with $Q(\lambda) = m$ gives a component of the preimage of $Z(m)$ in D_V and $\log\Big(\frac{m}{|Q(\lambda_x)|}\Big)$ measures how close the preimage of x^σ is to this component. Therefore $A(m,x)$ is indeed analogous to the local intersection number in the non-Archimedean case. The main term of $A(m,x)$ is bounded using the circle method in [SSTT22, Proposition 6.2]. Our calculation at the non-Archimedean place motivates our method to bound the error term of $A(m,x)$. Despite the fact that the same geometry underlies both the Archimedean and non-Archimedean cases, the Archimedean calculation is more subtle and involved than that in the non-Archimedean cases, and is carried out in [SSTT22, Proposition 6.4]. In sum, we prove that away from a thin set of bad m's, the Archimedean local contribution $\phi_m(x^\sigma) = o(m^{b/2}\log m)$, which combined with Equation (4.6.3), proves Theorem 4.3.

3. In order to treat the case of squares (which we need for Theorem 4.3 (2)), we need the level of control on $\mathrm{Disc}(L_n)$ given by Theorem 4.12. Indeed, to compare with the proof in §4.4.3, instead of making use of the special fact $\mathrm{rk}\,\Lambda \le 2$ in the $b = 2$ case to truncate the sum, we use Equation (4.6.4) and the arguments in Steps 2-3 to show that

$$\sum_{m=1}^{M} i_v(x, \mathcal{Z}(m)) = \sum_{n=1}^{[C_2 \log M]} \#\{\lambda \in L_n : Q(\lambda) \le M\} + O(X^{\frac{b+1}{2}})$$

and we conclude the proof of the local bound in [SSTT22, pp. 40-41] using Theorem 4.12.

4. In the characteristic p case, recall that $(C.\mathcal{Z}(m)) \asymp m^{b/2}$. Therefore, a bound of $O(M^{\frac{b+2}{2}})$ obtained in Step 3 above is not good enough to prove Theorem 4.4. Note that this only happens when $\mathrm{rk}\, L_n = b+2$, i.e., for local intersection multiplicities at supersingular points. However, note that the argument in Steps 2-3 shows that from (4.6.5), we have

$$\sum_{m=1}^{M} i_v(x, \mathcal{Z}(m)) = \frac{M^{(b+2)/2}}{a_{b+2}(n)} + O(M^{\frac{b+1}{2}}).$$

Therefore, we need to prove strong control on $a_{b+2}(n) \asymp D(L_n)$, analogous to the case of Theorem 4.2. This is carried out in [MST22b, Sections 5 and 6].

[43] Choose one preimage here. Note that the total sum $A(m,x)$ is independent of the choice.

5. An added complication in Theorem 4.4 that does not arise in the setting of Theorem 4.2 is that we need to obtain a tractable description of the F-crystal $\mathbf{L}_{\mathrm{cris}}$ in the formal neighbourhood of a supersingular but not superspecial point. To overcome this difficulty, we carefully analyze work of Ogus [Ogu79] where he characterizes F-crystals associated to supersingular K3 surfaces in terms of "characteristic subspaces".[44] This is carried out in [MST22b, Section 4].

4.7 The Hecke orbit conjecture

Let $\mathcal{S}'_{\mathbb{Q}} \subset \mathcal{A}_g$ be a Shimura variety of Hodge type with reflex field E. Let $\mathfrak{p}$ be a prime of E with residue characteristic p and we assume that $\mathcal{S}'_{\mathbb{Q}}$ has good reduction at $\mathfrak{p}$ and let $\mathcal{S}'$ denote the canonical integral model constructed in [Kis10]. We assume that the special fiber $S_{\mathbb{F}_{\mathfrak{p}}}$ intersects the ordinary locus of $\mathcal{A}_{g,\mathbb{F}_p}$. Define $\mathcal{S}'^{,\mathrm{ord}}_{\mathbb{F}_{\mathfrak{p}}}$ to be the intersection of $\mathcal{S}'_{\mathbb{F}_{\mathfrak{p}}}$ with the ordinary locus of $\mathcal{A}_{g,\mathbb{F}_p}$. The ordinary Hecke orbit conjecture of Chai–Oort posits that the prime-to-p Hecke orbit of any ordinary point $x \in \mathcal{S}'^{,\mathrm{ord}}_{\mathbb{F}_{\mathfrak{p}}}$ is Zariski dense in $\mathcal{S}'_{\mathbb{F}_{\mathfrak{p}}}$. In [Cha95], Chai proves that the prime-to-p Hecke orbit of an ordinary point $x \in \mathcal{A}_g(\overline{\mathbb{F}}_p)$ is Zariski dense. Chai's beautiful argument can broadly be summarized as follows:

1. Let $x \in \mathcal{A}_g(\overline{\mathbb{F}}_p)$ denote an ordinary point, such that the corresponding abelian variety A is isogenous to E^g where E is some elliptic curve. Using a careful analysis of the formal neighbourhood of ordinary points in terms of Serre–Tate coordinates, Chai proves that the prime-to-p Hecke orbit of any subvariety containing x must be Zariski dense. This step involves proving that any ordinary Hecke-stable subvariety of $\mathcal{A}_g$ must be "formally linear", namely its formal completion at any ordinary point must be a formal subtorus of the Serre–Tate torus. Chai also crucially uses the fact that $\mathrm{End}(A_{\overline{\mathbb{F}}_p}) \otimes \mathbb{Z}_p = \mathrm{End}(A_{\overline{\mathbb{F}}_p}[p^\infty])$ for A as above. Abelian varieties (and the associated moduli points) with this property are called *hypersymmetric*.

2. Chai makes the following observation: every abelian variety over $\overline{\mathbb{F}}_p$ has CM, and therefore also has real multiplication. Therefore, every $x \in \mathcal{A}_g$ must be contained in a Hilbert modular variety. Chai then proves the ordinary Hecke orbit conjecture for Hilbert modular varieties. The setting of Hilbert modular varieties is more tractable than the setting of $\mathcal{A}_g$ because *every point is hypersymmetric with respect to the Hilbert modular variety.* More precisely, let F denote the degree g totally real field defining the Hilbert modular variety, and let $A/\overline{\mathbb{F}}_p$ denote any abelian variety having real multiplication by F. Then, we have $\mathrm{End}_F(A) \otimes \mathbb{Q}_p = \mathrm{End}_F(A[p^\infty]) \otimes \mathbb{Q}_p$. Here, the subscript $_F$ refers to endomorphisms that commute with the real multiplication.

3. The previous step reduces the main theorem to proving that the prime-to-p Hecke orbit of a Hilbert modular variety is Zariski dense. But every Hilbert modular variety contains ordinary points isogenous to E^g! This concludes Chai's proof.

Our proof of Theorem 4.5 in the setting of the GSpin Shimura variety $\mathcal{S}$ is completely different from Chai's proof in the setting of $\mathcal{A}_g$. Let $Y \subset \mathcal{S}_{\mathbb{F}_p}$ denote a closed Hecke-stable generically ordinary subvariety. Our proof has the following outline. Using an argument of Chai ([Cha95, Section 2]) at the zero-dimensional cusp and the description of the toroidal compactification at one-dimensional

[44] In fact, Ogus' work can be used to give an explicit description of affine Deligne–Luztig varieties associated to orthogonal and Gspin Shimura varieties.

cusps of compactifications of $\mathcal{S}$, we prove that $Y = \mathcal{S}_{\mathbb{F}_p}$ if Y is not proper, i.e., if the closure of Y in the Baily–Borel compactification of $\mathcal{S}$ contains any boundary points.

Otherwise, Y must contain a proper generically ordinary curve $C \subset Y$ and by monodromy considerations, we may choose such a curve C which is not contained in any special divisor $\mathcal{Z}(m)$. By [MST22b, Theorem 1.2], we have that C, and therefore Y, must intersect $\mathcal{Z}(m)$ at an ordinary point for some m relatively prime to p.[45] We induct on the dimension of $\mathcal{S}$ (using the fact that $\mathcal{Z}(m)$ is a lower-dimensional Gspin Shimura variety) to show that Y must contain $\mathcal{Z}(m)_{\mathbb{F}_p}$. Then, monodromy considerations again imply that the prime-to-p Hecke orbit of $\mathcal{Z}(m)_{\mathbb{F}_p}$ is Zariski dense in $\mathcal{S}_{\mathbb{F}_p}$, whence the theorem follows. This entire argument is carried out in [MST22b, Section 8].

Remark 4.14. Since then, van Hoften has proved the ordinary Hecke orbit conjecture for Shimura varieties of Hodge type building on Chai's original method [vH23]. Crucial to van Hoften's proof is D'Addezio's proof of the parabolicity conjecture [D'A23]. Again using D'Addezio's work on the parabolicity conjecture, van Hoften and D'Addezio (in [DvH22]) also prove Chai–Oort's Hecke orbit conjecture for all Newton strata with mild restrictions on p.

4.8 Bad reduction

We now give a brief sketch of Tayou's proof [Tay24]. As explained before, when $b \leq 2$, either the Shimura variety is the product of modular curves treated in the work of Charles and Chai–Oort, or we can choose to only work with compact $\mathcal{Z}(m)$ in $\mathcal{S}$ and thus we do not need to worry about intersection happens at boundary. Thus we focus on the $b \geq 3$ case here. We consider C as before now in a toroidal compactification $\mathcal{S}^\Sigma$ with respect to a refined enough rational polyhedral cone decomposition (we will require this cone decomposition to not only satisfy the conditions in §3.6 but also some extra conditions depending on the generic point of C to be specified later).

4.8.1 Global intersection and Archimedean contribution

The same argument as in §4.6.2 using modularity results in §3.7 yields that $(C.(\mathcal{Z}^\Sigma(m), \phi_m)) \asymp m^{b/2}\log m$ for all m in the number field case and $(C.\mathcal{Z}^\Sigma(m)) \asymp m^{b/2}$ for all m in the characteristic p case. By [SSTT22, Prop. 5.4, Thm. 6.1] on the estimate of ϕ_m (see Remark 4.13 3)), we conclude that the sum of non-Archimedean local intersection multiplicities $(C.\mathcal{Z}^\Sigma(m)) \asymp m^{b/2}\log m$ for all m not in a set $S_{\mathrm{bad}} \subset \mathbb{Z}_{>0}$ of logarithmic asymptotic density zero.

Recall the definition of $\mathcal{Z}^\Sigma(m)$ from §3.7. Thus, we have (here we write the formula for the number field case and note that the characteristic p case is similar)

$$(C.\mathcal{Z}(m)) = (C.\mathcal{Z}(m)^\Sigma) - \sum_{J_\mathbb{Q}} \mathrm{mult}_{J_\mathbb{Q}}(m)(C.\mathcal{B}_{J_\mathbb{Q}}) - \sum_{I_\mathbb{Q},\mathbb{R}_{>0}w} \mathrm{mult}_{I_\mathbb{Q},\mathbb{R}_{>0}w}(C.\mathcal{B}_{I_\mathbb{Q},\mathbb{R}_{>0}w}).$$

By the definition of $\mathrm{mult}_{J_\mathbb{Q}}(m)$ in §3.7, since $J_L^\perp/J$ is a rank $b-2$ positive definite quadratic lattice, we have $\mathrm{mult}_{J_\mathbb{Q}}(m) \asymp m^{\frac{b}{2}-1+\epsilon} = o(m^{b/2})$ and thus $\sum_{J_\mathbb{Q}} \mathrm{mult}_{J_\mathbb{Q}}(m)(C.\mathcal{B}_{J_\mathbb{Q}}) \asymp m^{\frac{b}{2}-1+\epsilon} = o(m^{b/2})$ (see [Tay24, §4.5] for details). For $\mathrm{mult}_{I_\mathbb{Q},\mathbb{R}_{>0}w}$, [Tay24, Prop. 4.13, §4.6] showed that $\mathrm{mult}_{I_\mathbb{Q},\mathbb{R}_{>0}w} = m^{\frac{b-1}{2}+\epsilon} = o(m^{b/2})$ and thus $\sum_{I_\mathbb{Q},\mathbb{R}_{>0}w} \mathrm{mult}_{I_\mathbb{Q},\mathbb{R}_{>0}w}(C.\mathcal{B}_{I_\mathbb{Q},\mathbb{R}_{>0}w}) = o(m^{b/2})$. The proof can be compared with the estimate of $R_x(0,m)$ in [SSTT22, §§5.3, 5.3.1]. The proof ingredients include the

[45] In this paper, for simplicity, we outlined a proof on infinite intersection with $\cup_{m\in\mathbb{Z}_{>0}}\mathcal{Z}(m)$ and the same proof applies to $\cup_{m\in\mathbb{Z}_{>0},p\nmid m}\mathcal{Z}(m)$ and see [MST22b] for details.

explicit formula of regularized theta lifting as in [Bru02, Thm. 2.14] and an equidistribution result on $v \in K$ with $Q(v) = m$ (see [Tay24, Prop. 4.14] for the precise statement in this case). Therefore, we conclude that $(C.\mathcal{Z}(m)) \asymp m^{b/2} \log m$ for all m not in a set $S_{\text{bad}} \subset \mathbb{Z}_{>0}$ of logarithmic asymptotic density zero in the number field case and $(C.\mathcal{Z}(m)) \asymp m^{b/2}$ for all m in the characteristic p case.

4.8.2 local intersection at non-Archimedean places

The discussion in §4.6.2 holds verbatim in proving that the local intersection number $i_v(C.\mathcal{Z}(m))$ (in the function field case when v is not supersingular) at good reduction places is on average (over m) $o(C.\mathcal{Z}(m))$ and the total supersingular contribution of local intersection number is on average $\alpha(C.\mathcal{Z}(m)) + o(C.\mathcal{Z}(m))$ for some positive constant $\alpha < 1$ (since the entire supersingular locus lies in $\mathcal{S}_{\mathbb{F}_p}$). Therefore to finish the proof, one only needs to show that the local intersection number $i_v(C.\mathcal{Z}(m))$ at bad reduction places is on average $o(C.\mathcal{Z}(m))$ (see [Tay24, Props. 4.7, 4.12] for the precise statements).

We first discuss the case when the reduction of C at v lies in a boundary divisor whose admissible parabolic is the stabilizer of an isotropic plane $J_{\mathbb{Q}}$. The desired estimate of $i_v(C.\mathcal{Z}(m))$ is proved in [Tay24, §§5.1.1, 5.2.1, 2.2.4, 2.3.1, 2.4.1], which we now briefly describe. The formal completion of $\mathcal{Z}(m)$ along this boundary divisor consists of components indexed by $\{\lambda \in J_L^{\perp}/J \mid Q(\lambda) = m\}$ and each component admits an explicit description given in [Tay24, Prop. 2.4 and the paragraphs above]. We define the following analogy of the lattice of special endomorphism of the abelian variety mod t^n (here t denotes a uniformizer): let

$$L_n = \{\lambda \in J_L^{\perp}/J \mid C \bmod t^n \text{ lies in the component in the formal completion of } \mathcal{Z}(Q(\lambda)) \text{ associated to } \lambda\};$$

by definition, $\{L_n\}$ is a decreasing sequence of sets and $i_v(C.\mathcal{Z}(m))$ equals to, up to a multiple of constant, $\sum_{n=1}^{\infty} \#\{\lambda \in L_n \mid Q(\lambda) = m\}$.[46] Using the explicit description of the components in the formal completion of $\mathcal{Z}(m)$, Tayou proved that L_n's are decreasing lattices of the same rank. By definition, the rank is at most $b-2$ and hence the same argument in §4.6.2 applies to this sequence of lattices and obtain the desired bound on $i_v(C.\mathcal{Z}(m))$.

We now discuss the case when the reduction of C at v lies in a boundary stratum whose admissible parabolic is the stabilizer of an isotropic line $I_{\mathbb{Q}}$ (note that the boundary stratum in this case may not necessarily be a divisor). As in the previous case, if the special $\mathcal{Z}(m)$ hits this boundary stratum, then m is representable by (K, Q). The key difference is that (K, Q) is indefinite and we cannot directly copy the proof in the above case.[47] In order to obtain positive definite lattices as before, we need to work with a sufficiently refined Σ so that all the bad reduction points of C lie in a boundary divisor stratum. This is achieved in [Tay24, Prop. 4.2]. Briefly speaking, the formal completion of C at v provides a rational ray $\mathbb{R}_{>0}w$ in C_Φ and we just need to refine Σ to include all these finitely many rays (from the finitely many bad reduction places) and

[46] Unlike the good reduction case, the definition of L_n here is merely a reorganization of the local intersection number without using any moduli interpretation of $\mathcal{Z}(m)$. It may be possible to provide a moduli interpretation of $\mathcal{Z}(m)$ along the boundary and define L_n as the lattice of special endomorphisms of the log abelian variety using the work of Madapusi [MP19, §3] and the work of Kajiwara, Kato, and Nakayama [KKN], but we do not pursue this direction here.

[47] Although there are extra conditions depending on the rational polyhedral in the toroidal stratum representative, one does not expect to obtain a positive lattice in general. Consider for instance the smooth compactification of $\mathcal{A}_2$ given by the Deligne–Mumford compactification of $\mathcal{M}_2$. When the reduction is totally degenerate, every $\mathcal{Z}(m)$ hits this boundary stratum of totally degenerate points. Indeed, the corresponding quadratic form is $x^2 + yz$ of signature $(2, 1)$.

thus the reductions of C all lie in boundary divisors.[48] Now given a boundary divisor associated to $(I_{\mathbb{Q}}, \mathbb{R}_{>0}w)$, by [Tay24, Prop. 2.5], the formal completion of $\mathcal{Z}(m)$ along this boundary divisor consists of components indexed by the finite set $\{\lambda \in K \cap w^{\perp} \mid Q(\lambda) = m\}$ and each component admits an explicit description.[49] Once we have positive definite lattices of rank at most $b-1$, the rest of the argument is similar to the above case (see [Tay24, §§5.1.2, 5.2.2] for details).

4.9 Function fields: beyond the ordinary case

The non-ordinary settings of Theorems 4.2 and 4.4 are significantly more complicated. Indeed, the most general theorems are not even true! Here are two types of counter-examples:

1. Consider a one-parameter family A/C of supersingular abelian surfaces. Let X/C denote the family of Kummer K3 surfaces associated to A. The generic Picard rank of X is the same as the Picard rank of any specialization of X. More generally, consider any family of supersingular K3 surfaces. The generic Picard rank equals the Picard rank of any specialization.

2. Let $\mathcal{H}$ denote the mod p special fiber of a Hilbert modular surface associated to a real quadratic field split at p. Let $C \subset \mathcal{H}$ denote the non-ordinary locus, and let A/C denote the family of abelian surfaces parameterized by points of C. It is easy to see that only supersingular points of C parameterize split abelian surfaces. More generally, let $\mathcal{S}_{\mathbb{F}_p}$ denote the special fiber at p of the canonical integral model of a GSpin Shimura variety associated to a quadratic space that is split over $\mathbb{Q}_p$. Let $N \subset \mathcal{S}_{\mathbb{F}_p}$ denote the unique (closed) Newton stratum that contains the supersingular locus N_{ss} as a codimension 1 subvariety. Every special divisor $\mathcal{Z}(m)$ has the property that $\mathcal{Z}(m) \cap N$ is contained in N_{ss}! Therefore, let $C \subset \mathcal{S}_{\mathbb{F}_p}$ denote a curve whose generic point is contained in $N \setminus N_{\mathrm{ss}}$. Then, such a curve C is also a counter-example to Theorem 4.4!

These two counterexamples are the "obvious" obstructions to Theorems 4.2 and 4.4 being true in full generality. We strongly expect that these are the only obstructions. Indeed, Ruofan Jiang has proved the following theorem:

Theorem 4.15 ([Jia23]). *Let $C/\overline{\mathbb{F}}_p$ be a smooth irreducible quasi-projective curve with a finite map to $\mathcal{A}_2$ whose image is generically almost ordinary. Let $A/k(C)$ denote the abelian surface over the function field of C induced by this map. Suppose that A has no extra endomorphisms[50]. Then there are infinitely many points of C that parameterize non-simple abelian surfaces.*

In joint work with Ruofan Jiang, Davesh Maulik and Ziquan Yang, we are working on generalizing Theorem 4.15 to the case of non-ordinary K3 surfaces.

5 Related conjectures and results

5.1 S-integral theorems

In [BIR08], Baker, Ih, and Rumely prove the following beautiful result. Let E/K be an elliptic curve over a number field, and suppose that $P \in E(K)$ is a non-torsion point. Let S be any finite

[48] Using the terminology in [MP19, §§3.2, 3.3], these rational rays are exactly the monodromy operators.

[49] Compare to the discussion in [MP19, §3.3].

[50] This condition rules out the possibility of the image of C being contained in a Hilbert modular surface

set of places of K containing all the Archimedean places. Then there are only finitely many torsion points in $E(\overline{K})$ that are S-integral with respect to P. In other words, there are only finitely many integral torsion points in the affine curve $E \setminus \{P\}$.

A natural question is to replace the elliptic curve E by the j-line, and to replace P and torsion points by a K-rational point and CM points respectively. In other words, we expect the following conjecture to hold:

Conjecture 5.1. *Let $j_0 \in Y(1)(K)$ denote a fixed K-point, and let S be any finite set of places of K containing the places of bad reduction of j_0 and Archimedean places. Then there are only finitely many CM j-invariants $j_{CM} \in X(1)(\overline{K})$ S-integral with respect to j_0.*

In [Hab15], Habegger proves this conjecture when $j = 0$. In [HMRL21], the authors prove this conjecture when j is any CM j-invariant. The general case of this conjecture, however, is unsolved. An approach to this conjecture, which was already used in the [Hab15] and [HMRL21] in special cases, is very similar to Charles' intersection-theoretic setup on the modular curve. For simplicity, let $j_0 \in Y_1(\mathbb{Z})$ and let $S = \{\infty, p\}$ where p is a single prime. Then the conjecture reduces to proving that given any CM j-invariant j_{CM} outside a finite set, there exists a prime $\ell \neq p$ such that $j_0 \equiv j_{CM} \bmod v$ where v is a non-Archimedean place of $\mathbb{Q}(j_{CM})$ dividing ℓ. Setting Archimedean places aside, Charles' setup reduces the conjecture to proving that $i_p(j_0, j_i) = o((j_0, j_i))$ where $\{j_i\}_{i\in\mathbb{Z}_{>0}}$ is the set of all CM j-invariants, (j_0, j_i) is the Arakelov intersection of j and j_i, and $i_p(j, j_i)$ is the sum of the v-adic contributions over the places v of $\mathbb{Q}(j_i)$ dividing p. This is exactly the p-adic transcendence question alluded to in Section 4.6. This question can also be posed in the setting of higher dimensional GSpin Shimura varieties, with $\mathcal{Z}(m)$ taking the place of CM j-invariants. Our methods are able to prove these bounds on average over m, but we would need these bounds for individual m to prove this conjecture.

This question can also be posed in the characteristic p case. Ruofan Jiang proves this conjecture in the setting of the almost-ordinary locus of $\mathcal{A}_2$ in [Jia23].

5.2 AIM conjecture

In a recent AIM workshop, a group consisting of the authors, Luis García, Debanjana Kundu, Lucia Mocz, Congling Qiu, Ari Shnidman, Salim Tayou, Yujie Xu, and Shouwu Zhang made the following conjecture (that massively generalizes the work of [Cha18], [CO06], [ST20], [SSTT22], [MST22], [MST22b], [Tay24] and [Jia23]):

Conjecture 5.2. *Let $\mathcal{S}$ denote the canonical integral model of a Shimura variety and let $X, Y \subset \mathcal{S}$ denote subschemes. We will assume that either X and Y are generically ordinary irreducible subvarieties of $\mathcal{S}_{\mathbb{F}_\mathfrak{p}}$ where $\mathfrak{p}$ is a prime (of the reflex field E) of good reduction for $\mathcal{S}$ and that* $\dim X + \dim Y = \dim \mathcal{S}_E$*, or we will assume that X and Y are flat over $\mathbb{Z}[1/N]$ (for a large enough integer N), and that* $\dim X_\mathbb{Q} + \dim Y_\mathbb{Q} = \dim \mathcal{S}_E - 1$*. In either case, X and Y have complementary dimension when thought of as arithmetic subschemes of $\mathcal{S}$. Then the set $\{(x, y) \in X \times Y : x \in \bigcup_T T(y)\}$, where T runs through Hecke correspondences on $\mathcal{S}$, is Zariski dense in $X \times Y$.*

We let $\mathcal{S}$ be a GSpin Shimura variety as in prior sections. By setting $X = C$ (or $X = \operatorname{Spec} \mathcal{O}_K[1/N]$), and $Y = \mathcal{Z}(m)_{\mathbb{F}_p}$ (or $Y = \mathcal{Z}(m)$), Conjecture 5.2 implies all the main results of *loc. cit.*. We will now briefly explain how the results of [Cha18],[ST20], [SSTT22] and [Tay24] implies Conjecture 5.2 when $X = \operatorname{Spec} \mathcal{O}_K[1/N]$ and $Y = \mathcal{Z}(m)$. Set $Z_1 := \{(x, y) \in X \times Y : x \in$

$\bigcup_T T(y)\}$,[51] and let $\mathrm{pr}_X, \mathrm{pr}_Y$ denote the projections of $X \times Y$ onto X and Y respectively. The main theorems in [Cha18, ST20, SSTT22, Tay24] imply that $\# \mathrm{pr}_X(Z_1) = \infty$, and therefore we have that $\mathrm{pr}_X(Z)$ is a non-empty Zariski open subscheme of X. Note that Y is a special divisor, and therefore a GSpin Shimura variety in its own right, and thus is equipped with its own set of Hecke correspondences (we remark that these Hecke correspondences induce Hecke correspondences on $\mathcal{S}$, and that these give a subset of the set of Hecke correspondences on $\mathcal{S}$). By definition, Z_1 is stable under the Hecke correspondences on Y (where these Hecke correspondences act trivially on the first coordinate, and the usual way on the second coordinate). Therefore, Z, the Zariski closure of Z_1, is also stable under these Hecke correspondences. We now pick a place v contained in $\mathrm{pr}_X(Z)$ such that $X_v = X \bmod v$ is ordinary.[52] The ordinary Hecke orbit conjecture applied to $Y_v = Y \bmod v$ yields that $\mathrm{pr}_Y Z$ must contain Y_v for all such primes. Therefore, we have an infinite set of places v of K such that Z contains $X_v \times Y_v$. It suffices to show that the fibers of the map $Z \to X$ generically have dimension $\dim Y$. The map $Z \to X$ is generically flat, and therefore by replacing X with a Zariski open subscheme (and Z with its restriction to the preimage of this Zariski open subscheme), we may assume that $Z \to X$ is faithfully flat. For any place v of X, the dimension of Z_v is the same as the dimension of $Z_{\mathbb{Q}} \to X_{\mathbb{Q}}$. The result now follows from the infinitude of ordinary places v and the observation that the fiber of Z over such places v is just $\mathrm{pr}_Y(Z_v) = Y_v$.

Now, consider the case when $\mathcal{S}$ is as above, $X = C \subset \mathcal{S}_{\overline{\mathbb{F}}_p}$ is a generically ordinary curve, and $Y = \mathcal{Z}(m)_{\overline{\mathbb{F}}_p}$. Defining Z_1 analogously, the above argument would apply verbatim if we knew that $\mathrm{pr}_X(Z_1)$ contained a Zariski dense set of X. This follows from the main results of [MST22] and [Tay24] when $\mathcal{S} = \mathcal{A}_2$, and we expect this to be the case in general (using the ideas of [MST22, MST22b, Tay24]).

Charles explained to us an inductive argument for arbitrary X and Y being a special cycle associated to a positive definite sublattice of L over $\mathbb{Z}[1/N]$ in a GSpin Shimura variety associated to a quadratic lattice L to show that $\# \mathrm{pr}_X(Z_1) = \infty$ using a theorem of Green (see for instance [Voi03, Prop. 17.20]) and main theorems in [ST20, SSTT22, Tay24] for $X = \operatorname{Spec} \mathcal{O}_K[1/N]$ case. Indeed, his idea combined with our argument above can show that for these X and Y, we have $Z = X \times Y$ as follows. From the same dimension argument above, we only need to show that there exists a Zariski dense subset $X_1 \subset X$ such that $\forall x \in X_1$, we have $\{x\} \times Y_{k(x)} \subset Z$. For the ease of exposition,[53] we directly use [TT23, Thm. 1.7] to conclude that there exists a Zariski dense subset $X_2 \subset X_{\mathbb{Q}}$ such that $\forall x \in X_2$, we have $x \in Y'(x)$, where $Y'(x)$ is a special cycle of $\mathcal{S}$ such that $\dim Y'(x)_{\mathbb{Q}} = \dim Y_{\mathbb{Q}} + 1$ and $Y'(x)$ contains a Hecke translate of Y. Then by applying our proof above for Conjecture 5.2 to the extension $\widetilde{x}$ of x over $\operatorname{Spec} \mathbb{Z}[1/N]$ in the GSpin Shimura variety $Y'(x)$, we conclude that $\{\widetilde{x}\} \times Y \subset Z$. We then obtain Conjecture 5.2 for X and Y by taking $X_1 = \{\widetilde{x} \mid x \in X_2\}$.

Unitary Shimura varieties with signature $(n, 1)$ are another class of Shimura varieties with special divisors. All the above mentioned theorems also hold for these unitary Shimura varieties. See [SSTT22, §9.3] and [MST22b, Rmk. 8.12].

[51] The precise meaning is that for each Hecke correspondence T on $\mathcal{S}_{\mathbb{Q}}$, we only consider it defined over $\mathcal{S}$ away from primes dividing the degree of T; this convention also applies later to Hecke correspondences on the Shimura subvariety Y.

[52] See the work of Joshi–Rajan [JR01], Bogomolov–Zarhin [BZ09], and unpublished work of Sawin [Saw16b] on density 1 of ordinary reductions of a K3 surface over a number field (after suitable finite field extension); their proofs apply to Kuga–Satake abelian varieties.

[53] Since we only need a Zariski dense subset of X with the desired propoerty (no requirement for equidistribution properties), we may argue inductively on dimension of X using Green's theorem following Charles's idea.

When neither X nor Y is special, Tayou and Tholozan [TT23] prove the complex-analytic setting of this conjecture in full generality. Asvin G. uses Chai–Oort's work to prove this conjecture when $\mathcal{S}$ is a product of modular curves in [G22]. In [GHS22], Asvin G. and Qiao He, and the first-named author prove this conjecture for mod p Hilbert modular surfaces associated to real quadratic fields split at p by observing that these surfaces admit a local product structure which facilitates the use of Chai–Oort's ideas from [CO06].

5.3 Generalization of Elkies theorem

Elkies proved in [Elk87] [Elk89] that any elliptic curve E over a number field K with at least one real embedding has infinitely many primes of supersingular reduction. In his proof, Elkies considered the 1-cycle associated to E in the j-line over $\mathbb{Z}$ and reached the conclusion by studying its intersections with well-chosen CM cycles. Later on, there are generalizations of Elkies's theorem to certain abelian surfaces A over $\mathbb{Q}$ with quaternionic multiplication (i.e., such an abelian surface admits infinitely many supersingular reductions). See the work of Jao [Jao03], Sadykov [Sad04], and Baba–Granath [BG08]. We refer to the readers for the precise theorems in their papers, but remark that all the abelian surfaces that they consider are parametrized by Shimura curves associated to some quaternion algebras over $\mathbb{Q}$ which are of genus 0 with suitable level structure. One may use Lang–Trotter heuristics and its higher dimensional generalizations to see that in the above cases for $E/\mathbb{Q}$ and $A/\mathbb{Q}$ with no more extra endomorphisms, the number of supersingular primes less than X should equal $X^{\frac{1}{2}\pm\epsilon}$ for any $\epsilon \geq 0$. In particular, the set of supersingular primes is known to be density 0, which is proved in the work of Serre [Ser13, #133][Ser81], Katz and Ogus [Ogu82, Prop. 2.7, Cor. 2.9], Sawin [Saw16].

More generally, one may expect that for an abelian variety A over $\mathbb{Q}$ parametrized by a Shimura curve has infinitely many non-ordinary reductions. Note that the reflex field of a Shimura curve in general may not be $\mathbb{Q}$ and thus the open Newton strata at many primes may not be ordinary and one refers to the open Newton stratum as the μ-ordinary one for a given prime. The natural expectation would be that A admits infinitely many non-μ-ordinary reductions. In the joint work in progress of Li, Mantovan, Pries, and the second-named author [LMPT], we verify this expectation when A is an abelian fourfold parametrized by one certain PEL type unitary Shimura curve. Note that although the reflex field of the Shimura curve is not $\mathbb{Q}$, it admits a natural model over $\mathbb{Q}$ and has a suitable level structure such that the curve is of genus 0. If A does not have extra endomorphisms (than the ones from being on the PEL Shimura curve), in the joint work in progress of Cantoral Farfán, Li, Mantovan, Pries and the second-named author [CFLM$^+$], we proved that the set of non-μ-ordinary primes of A is of density 0.[54]

Note that all the cases above have the corresponding Shimura curve to be genus 0. To compare with theorems in §4.1, from the heuristics aspect (see §2.1), both settings are about infinite intersection of the arithmetic 1-cycle given by A and union of divisors in special fibers of the corresponding Shimura variety; the key difference is that in theorems in §4.1, the divisors in special fibers are all from reductions of special divisors $\mathcal{Z}(m)$, while in the attempts to generalize Elkies theorem, one needs to work with the non-μ-ordinary locus in each special fiber, which is varying from prime to prime. In the modular curve case, Elkies's strategy is to use CM cycles, which have supersingular reductions at half of primes, and prove that one can find intersection with the CM cycles at one of the supersingular primes; similar strategy is used in all generalizations mentioned

[54]See also [Saw16b] for related results.

above. Our proofs of theorems in §4.1 do not provide information as in Elkies's strategy on where the intersection happens (to our knowledge, the information one may obtain from our proofs is a very loose estimate on the approximate size of the primes). The genus 0 property is a key input in Elkies's type strategy.

5.4 Upper bounds for certain reduction types

The theorems in §4.1, Elkies's theorem and its generalizations discussed in §5.3 are lower bounds on the set of primes modulo which A lie in certain isogeny classes for a given abelian variety A over a number field. There are many more work on upper bounds and we would not be able to provide a complete list. We only provide a few examples in addition to some of the density 0 of the set of non-ordinary reductions mentioned in §5.3. We would like to refer the reader to the papers cited below and their references for a more comprehensive history in these directions.

Zywina [Zyw14, Cor. 1.3] proves that if one assumes the Mumford–Tate conjecture for an abelian variety A/K,[55] the set of primes v such that $A_{\overline{\mathbb{F}}_v}$ is not simple is of density 0 (after possibly replacing K by a finite extension) if $\mathrm{End}(A_{\overline{K}})$ is commutative. Indeed, Zywina's proof provides an explicit upper bound. The unconditional statement was previously conjectured by Murty and Patankar [MP08], and some special cases of this conjecture were also previously proved in the work of Chavdarov [Cha97] and Achter [Ach09, Ach12] (both provide explicit bounds). In the case when A is an abelian surface, the bounds have been improved by recent work of Wang [Wan23].

The Lang–Trotter philosophy and its higher dimensional generalizations can be used to make many other conjectures on the behavior of Frobenius traces. There are several results in this direction — see for instance the work of Katz [Kat09, §1] and the work of Cojocaru, Davis, Silverberg and Stange [CDSS17]. In [CDSS17], the authors provide an upper bound for the set of primes with certain Frobenius trace for a given generic abelian variety over $\mathbb{Q}$ towards the Lang–Trotter style conjecture they proposed and this bound has been improved in recent work of Cojocaru and Wang [CW22]. One may also consider the same question for abelian varieties which are not generic - see for instance another recent work of Cojocaru and Wang [CW23].

References

[Ach09] Jeffrey D. Achter, *Split reductions of simple abelian varieties*, Math. Res. Lett. **16** (2009), no. 2, 199–213, DOI 10.4310/MRL.2009.v16.n2.a1. MR2496739

[Ach12] ———, *Explicit bounds for split reductions of simple abelian varieties*, J. Théor. Nombres Bordeaux **24** (2012), no. 1, 41–55 (English, with English and French summaries). MR2914900

[AH17] Jeffrey D. Achter and Everett W. Howe, *Split abelian surfaces over finite fields and reductions of genus-2 curves*, Algebra Number Theory **11** (2017), no. 1, 39–76.

[AGHMP17] Fabrizio Andreatta, Eyal Z. Goren, Benjamin Howard, and Keerthi Madapusi Pera, *Height pairings on orthogonal Shimura varieties*, Compos. Math. **153** (2017), no. 3, 474–534.

[AGHMP18] ———, *Faltings heights of abelian varieties with complex multiplication*, Ann. of Math. (2) **187** (2018), no. 2, 391–531.

[AMRT10] Avner Ash, David Mumford, Michael Rapoport, and Yung-Sheng Tai, *Smooth compactifications of locally symmetric varieties*, 2nd ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2010. With the collaboration of Peter Scholze.

[55] The Mumford–Tate conjecture holds for many abelian varieties — see for instance the work of Pink [Pin98] and Vasiu [Vas08] and references there.

[Aut03] Pascal Autissier, *Hauteur des correspondances de Hecke*, Bull. Soc. Math. France **131** (2003), no. 3, 421–433 (French, with English and French summaries).

[Aut05] ———, *Hauteur moyenne de variétés abéliennes isogènes*, Manuscripta Math. **117** (2005), no. 1, 85–92 (French, with English and French summaries).

[BG08] Srinath Baba and Håkan Granath, *Primes of superspecial reduction for QM abelian surfaces*, Bull. Lond. Math. Soc. **40** (2008), no. 2, 311–318.

[BIR08] Matthew Baker, Su-ion Ih, and Robert Rumely, *A finiteness property of torsion points*, Algebra Number Theory **2** (2008), no. 2, 217–248, DOI 10.2140/ant.2008.2.217. MR2377370

[Bha] Tejasi Bhatnagar, *Size of isogeny classes of abelian varieties in the extended Lubin-Tate case*. In preparation.

[BZ09] Fedor A. Bogomolov and Yuri G. Zarhin, *Ordinary reduction of $K3$ surfaces*, Cent. Eur. J. Math. **7** (2009), no. 2, 206–213, DOI 10.2478/s11533-009-0013-8. MR2506961

[Bor98] Richard E. Borcherds, *Automorphic forms with singularities on Grassmannians*, Invent. Math. **132** (1998), no. 3, 491–562.

[BY23] Daniel Bragg and Ziquan Yang, *Twisted derived equivalences and isogenies between K3 surfaces in positive characteristic*, Algebra Number Theory **17** (2023), no. 5, 1069–1126, DOI 10.2140/ant.2023.17.1069. MR4585353

[Bru02] Jan H. Bruinier, *Borcherds products on O(2, l) and Chern classes of Heegner divisors*, Lecture Notes in Mathematics, vol. 1780, Springer-Verlag, Berlin, 2002.

[BBGK07] Jan H. Bruinier, José I. Burgos Gil, and Ulf Kühn, *Borcherds products and arithmetic intersection theory on Hilbert modular surfaces*, Duke Math. J. **139** (2007), no. 1, 1–88.

[BHK+20] Jan H. Bruinier, Benjamin Howard, Stephen S. Kudla, Michael Rapoport, and Tonghai Yang, *Modularity of generating series of divisors on unitary Shimura varieties*, Astérisque **421, Diviseurs arithmétiques sur les variétés orthogonales et unitaires de Shimura** (2020), 7–125.

[BK03] Jan Hendrik Bruinier and Ulf Kühn, *Integrals of automorphic Green's functions associated to Heegner divisors*, Int. Math. Res. Not. **31** (2003), 1687–1729.

[BK01] Jan Hendrik Bruinier and Michael Kuss, *Eisenstein series attached to lattices and modular forms on orthogonal groups*, Manuscripta Math. **106** (2001), no. 4, 443–459.

[BY09] Jan Hendrik Bruinier and Tonghai Yang, *Faltings heights of CM cycles and derivatives of L-functions*, Invent. Math. **177** (2009), no. 3, 631–681.

[BZ21] Jan Hendrik Bruinier and Shaul Zemel, *Special cycles on toroidal compactifications of orthogonal Shimura varieties*, Math. Ann. (2021). published online.

[BGKK07] J. I. Burgos Gil, J. Kramer, and U. Kühn, *Cohomological arithmetic Chow rings*, J. Inst. Math. Jussieu **6** (2007), no. 1, 1–172.

[CFLM+] Victoria Cantoral Farfán, Wanlin Li, Elena Mantovan, Rachel Pries, and Yunqing Tang, *Positive density of primes of ordinary reduction for abelian varieties of simple signature*. In preparation.

[Cha95] Ching-Li Chai, *Every ordinary symplectic isogeny class in positive characteristic is dense in the moduli*, Invent. Math. **121** (1995), no. 3, 439–479.

[CO06] Ching-Li Chai and Frans Oort, *Hypersymmetric abelian varieties*, Pure Appl. Math. Q. **2** (2006), no. 1, Special Issue: In honor of John H. Coates., 1–27.

[Cha14] François Charles, *On the Picard number of K3 surfaces over number fields*, Algebra Number Theory **8** (2014), no. 1, 1–17.

[Cha16] ———, *Birational boundedness for holomorphic symplectic varieties, Zarhin's trick for $K3$ surfaces, and the Tate conjecture*, Ann. of Math. (2) **184** (2016), no. 2, 487–526, DOI 10.4007/annals.2016.184.2.4. MR3548531

[Cha18] ———, *Exceptional isogenies between reductions of pairs of elliptic curves*, Duke Math. J. **167** (2018), no. 11, 2039–2072.

[Cha97] Nick Chavdarov, *The generic irreducibility of the numerator of the zeta function in a family of curves with large monodromy*, Duke Math. J. **87** (1997), no. 1, 151–180.

[COU01] Laurent Clozel, Hee Oh, and Emmanuel Ullmo, *Hecke operators and equidistribution of Hecke points*, Invent. Math. **144** (2001), no. 2, 327–351.

[CDSS17] Alina Carmen Cojocaru, Rachel Davis, Alice Silverberg, and Katherine E. Stange, *Arithmetic properties of the Frobenius traces defined by a rational abelian variety (with two appendices by J-P. Serre)*, Int. Math. Res. Not. IMRN **12** (2017), 3557–3602, DOI 10.1093/imrn/rnw058. MR3693659

[CW22] Alina Carmen Cojocaru and Tian Wang, *Bounds for the distribution of the Frobenius traces associated to a generic abelian variety* (2022). arXiv:2207.02913.

[CW23] ______, *Bounds for the distribution of the Frobenius traces associated to products of non-CM elliptic curves*, Canad. J. Math. **75** (2023), no. 3, 687–712, DOI 10.4153/S0008414X22000086. MR4586829

[D'A23] Marco D'Addezio, *Parabolicity conjecture of F-isocrystals*, Ann. of Math. (2) **198** (2023), no. 2, 619–656.

[DvH22] Marco D'Addezio and Pol van Hoften, *Hecke orbits on Shimura varieties of Hodge type* (2022). arXiv:2205.10344.

[Elk87] Noam D. Elkies, *The existence of infinitely many supersingular primes for every elliptic curve over* $\mathbf{Q}$, Invent. Math. **89** (1987), no. 3, 561–567, DOI 10.1007/BF01388985.

[Elk89] ______, *Supersingular primes for elliptic curves over real number fields*, Compositio Math. **72** (1989), no. 2, 165–172.

[EGT] Philip Engel, François Greer, and Salim Tayou, *Mixed Mock modularity of special divisors.* arXiv:2301.05982.

[EK95] Alex Eskin and Yonatan R. Katznelson, *Singular symmetric matrices*, Duke Math. J. **79** (1995), no. 2, 515–547, DOI 10.1215/S0012-7094-95-07913-7.

[FC90] Gerd Faltings and Ching-Li Chai, *Degeneration of abelian varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 22, Springer-Verlag, Berlin, 1990. With an appendix by David Mumford.

[Fu23] Yu Fu, *Isogeny classes of non-simple abelian surfaces over finite fields* (2023). arXiv:2308.11132.

[G22] Asvin G, *Unlikely and just likely intersections for high dimensional families of elliptic curves.* (2022). arXiv:2203.16420.

[GHS22] Asvin G, Qiao He, and Ananth N. Shankar, *Just-likely intersections on Hilbert modular surfaces.* (2022). arXiv:2209.02806.

[Hab15] Philipp Habegger, *Singular moduli that are algebraic units*, Algebra Number Theory **9** (2015), no. 7, 1515–1524, DOI 10.2140/ant.2015.9.1515. MR3404647

[Han04] Jonathan Hanke, *Local densities and explicit bounds for representability by a quadratric form*, Duke Math. J. **124** (2004), no. 2, 351–388.

[HZ01] Michael Harris and Steven Zucker, *Boundary cohomology of Shimura varieties. III. Coherent cohomology on higher-rank boundary strata and applications to Hodge theory*, Mém. Soc. Math. Fr. (N.S.) **85** (2001), vi+116 (English, with English and French summaries).

[HMRL21] Sebastian Herrero, Ricardo Menares, and Juan Rivera-Letelier, *There are at most finitely many singular moduli that are S-units.* (2021). To appear in Compositio.

[HP20] Benjamin Howard and Keerthi Madapusi Pera, *Arithmetic of Borcherds products*, Astérisque **421, Diviseurs arithmétiques sur les variétés orthogonales et unitaires de Shimura** (2020), 187–297.

[IIK21] Kazuhiro Ito, Tetsushi Ito, and Teruhisa Koshikawa, *CM liftings of $K3$ surfaces over finite fields and their applications to the Tate conjecture*, Forum Math. Sigma **9** (2021), Paper No. e29, 70, DOI 10.1017/fms.2021.24. MR4241794

[Jao03] David Yen Jao, *Supersingular primes for rational points on modular curves*, ProQuest LLC, Ann Arbor, MI, 2003. Thesis (Ph.D.)–Harvard University.

[Jia23] Ruofan Jiang, *Almost ordinary Abelian surfaces over global function fields with application to integral points* (2023). arXiv:2304.07715.

[JR01] Kirti Joshi and C. S. Rajan, *Frobenius splitting and ordinarity* (2001). arXiv:math/0110070.

[KKN] Takeshi Kajiwara, Kazuya Kato, and Chikara Nakayama, *Moduli of logarithmic abelian varieties with PEL structure.* arXiv:2205.10985.

[Kat09] Nicholas M. Katz, *Lang-Trotter revisited*, Bull. Amer. Math. Soc. (N.S.) **46** (2009), no. 3, 413–457, DOI 10.1090/S0273-0979-09-01257-9. MR2507277

[KMP16] Wansu Kim and Keerthi Madapusi Pera, *2-adic integral canonical models*, Forum Math. Sigma **4** (2016), Paper No. e28, 34, DOI 10.1017/fms.2016.23. MR3569319

[Kis10] Mark Kisin, *Integral models for Shimura varieties of abelian type*, J. Amer. Math. Soc. **23** (2010), no. 4, 967–1012.

[LMPT] Wanlin Li, Elena Mantovan, Rachel Pries, and Yunqing Tang, *Basic reductions of Jacobians of certain cyclic covers of the projective line*. In preparation.

[MP15] Keerthi Madapusi Pera, *The Tate conjecture for K3 surfaces in odd characteristic*, Invent. Math. **201** (2015), no. 2, 625–668, DOI 10.1007/s00222-014-0557-5. MR3370622

[MP16] ———, *Integral canonical models for spin Shimura varieties*, Compos. Math. **152** (2016), no. 4, 769–824.

[MP19] ———, *Toroidal compactifications of integral models of Shimura varieties of Hodge type*, Ann. Sci. Éc. Norm. Supér. (4) **52** (2019), no. 2, 393–514 (English, with English and French summaries).

[MP20] ———, *Erratum to appendix to '2-adic integral canonical models'*, Forum Math. Sigma **8** (2020), Paper No. e14, 11, DOI 10.1017/fms.2020.2. MR4076653

[Mau14] Davesh Maulik, *Supersingular K3 surfaces for large primes*, Duke Math. J. **163** (2014), no. 13, 2357–2425, DOI 10.1215/00127094-2804783. With an appendix by Andrew Snowden. MR3265555

[MST20] Davesh Maulik, Ananth N. Shankar, and Yunqing Tang, *Reductions of abelian surfaces over global function fields* (2020). arXiv:1812.11679.

[MST22a] ———, *Reductions of abelian surfaces over global function fields*, Compos. Math. **158** (2022), no. 4, 893–950.

[MST22b] ———, *Picard ranks of K3 surfaces over function fields and the Hecke orbit conjecture*, Invent. Math. **228** (2022), no. 3, 1075–1143.

[MP08] V. Kumar Murty and Vijay M. Patankar, *Splitting of abelian varieties*, Int. Math. Res. Not. IMRN **12** (2008), Art. ID rnn033, 27, DOI 10.1093/imrn/rnn033.

[NO85] Niels Nygaard and Arthur Ogus, *Tate's conjecture for $K3$ surfaces of finite height*, Ann. of Math. (2) **122** (1985), no. 3, 461–507, DOI 10.2307/1971327. MR0819555

[Ogu79] Arthur Ogus, *Supersingular $K3$ crystals*, Journées de Géométrie Algébrique de Rennes (Rennes, 1978), Astérisque, vol. 64, Soc. Math. France, Paris, 1979, pp. 3–86.

[Ogu82] ———, *Hodge cycles and crystalline cohomology*, Hodge cycles, motives, and Shimura varieties, Lecture Notes in Mathematics, vol. 900, Springer-Verlag, Berlin-New York, 1982, pp. 357–414.

[Pin90] Richard Pink, *Arithmetical compactification of mixed Shimura varieties*, Bonner Mathematische Schriften [Bonn Mathematical Publications], vol. 209, Universität Bonn, Mathematisches Institut, Bonn, 1990. Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, 1989.

[Pin98] ———, *l-adic algebraic monodromy groups, cocharacters, and the Mumford-Tate conjecture*, J. Reine Angew. Math. **495** (1998), 187–237, DOI 10.1515/crll.1998.018. MR1603865

[Qiu] Congling Qiu, *Modularity of arithmetic special divisors for unitary Shimura varieties*. with an appendix by Yujie Xu; arXiv:2204.13457.

[Sad04] Marat Sadykov, *Two results in the arithmetic of Shimura curves*, ProQuest LLC, Ann Arbor, MI, 2004. Thesis (Ph.D.)–Columbia University.

[Saw16a] William F. Sawin, *Ordinary primes for Abelian surfaces*, C. R. Math. Acad. Sci. Paris **354** (2016), no. 6, 566–568, DOI 10.1016/j.crma.2016.01.025 (English, with English and French summaries). MR3494322

[Saw16b] ———, *The distribution of the Newton polygon of a K3 surface* (2016). unpublished notes.

[Ser81] Jean-Pierre Serre, *Quelques applications du théorème de densité de Chebotarev*, Inst. Hautes Études Sci. Publ. Math. **54** (1981), 323–401 (French). MR644559

[Ser13] ———, *Oeuvres/Collected papers. IV. 1985–1998*, Springer Collected Works in Mathematics, Springer, Heidelberg, 2013 (French). Reprint of the 2000 edition [MR1730973]. MR3185222

[SSTT22] Ananth N. Shankar, Arul Shankar, Yunqing Tang, and Salim Tayou, *Exceptional jumps of Picard ranks of reductions of K3 surfaces over number fields*, Forum Math. Pi **10** (2022), Paper No. e21, 49, DOI 10.1017/fmp.2022.14. MR4490194

[ST20] Ananth N. Shankar and Yunqing Tang, *Exceptional splitting of reductions of abelian surfaces*, Duke Math. J. **169** (2020), no. 3, 397–434.

[ST18] Ananth N. Shankar and Jacob Tsimerman, *Unlikely intersections in finite characteristic*, Forum Math. Sigma **6** (2018), Paper No. e13, 17, DOI 10.1017/fms.2018.15. MR3841493

[Tan90] S. G. Tankeev, *Surfaces of $K3$ type over number fields and the Mumford-Tate conjecture*, Izv. Akad. Nauk SSSR Ser. Mat. **54** (1990), no. 4, 846–861 (Russian); English transl., Math. USSR-Izv. **37** (1991), no. 1, 191–208.

[Tan95] ———, *Surfaces of $K3$ type over number fields and the Mumford-Tate conjecture. II*, Izv. Ross. Akad. Nauk Ser. Mat. **59** (1995), no. 3, 179–206 (Russian, with Russian summary); English transl., Izv. Math. **59** (1995), no. 3, 619–646.

[Tay24] Salim Tayou, *Picard rank jumps for K3 surfaces with bad reduction.* (2024). arXiv:2203.09559, to appear in Algebra and Number Theory.

[TT23] Salim Tayou and Nicolas Tholozan, *Equidistribution of Hodge loci II*, Compos. Math. **159** (2023), no. 1, 1–52, DOI 10.1112/s0010437x22007795. MR4533443

[vH23] Pol van Hoften, *On the ordinary Hecke orbit conjecture* (2023). arXiv:2112.12422, to appear in Algebra and Number Theory.

[Vas08] Adrian Vasiu, *Some cases of the Mumford-Tate conjecture and Shimura varieties*, Indiana Univ. Math. J. **57** (2008), no. 1, 1–75.

[Voi03] Claire Voisin, *Hodge theory and complex algebraic geometry. II*, Cambridge Studies in Advanced Mathematics, vol. 77, Cambridge University Press, Cambridge, 2003. Translated from the French by Leila Schneps. MR1997577

[Wan23] Tian Wang, *Distribution of primes of split reductions for abelian surfaces* (2023). arXiv:2205.15199.

[Yan10] Tonghai Yang, *An arithmetic intersection formula on Hilbert modular surfaces*, Amer. J. Math. **132** (2010), no. 5, 1275–1309.

[Zar83] Yu. G. Zarhin, *Hodge groups of $K3$ surfaces*, J. Reine Angew. Math. **341** (1983), 193–220.

[Zha22] Shou-Wu Zhang, *Standard conjectures and height pairings*, Pure Appl. Math. Q. **18** (2022), no. 5, 2221–2278.

[Zyw14] David Zywina, *The splitting of reductions of an abelian variety*, Int. Math. Res. Not. IMRN **18** (2014), 5042–5083. MR3264675