Linear transformations

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$T(\underline{v}) = \underline{w}$, function vector to vector

Def. T is linear if:

1) $T(\underline{v}_1 + \underline{v}_2) = T(\underline{v}_1) + T(\underline{v}_2)$

2) $T(c\underline{v}) = cT(\underline{v})$

i.e. Can move $+$ and c in and out of brackets

eg: $T(\underline{v}) = \underline{v} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $T : \mathbb{R}^2 \to \mathbb{R}^2$

$$T(\underline{v}_1 + \underline{v}_2) = \underline{v}_1 + \underline{v}_2 + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$T(\underline{v}_1) + T(\underline{v}_2) = \underline{v}_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \underline{v}_2 + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow T(\underline{v}_1 + \underline{v}_2) \neq T(\underline{v}_1) + T(\underline{v}_2)$$

$\rightarrow$ not linear

Matrix of linear transformation

Def. Elementary vectors

e.g. $\mathbb{R}^2$, $\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\underline{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

e.g. $\mathbb{R}^3$, $\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$,

$\vdots$

e.g. $\mathbb{R}^n$, $\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\ldots$, $\underline{e}_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

Given $T: \mathbb{R}^n \to \mathbb{R}^m$, matrix $\underline{\underline{A}}$ of transformation

$$\underline{\underline{A}} = \left[T(\underline{e}_1) \ \dots \ T(\underline{e}_n) \right]$$

$m \times n$ matrix

e.g. $T(\underline{v}) = 2\underline{v}$, $T: \mathbb{R}^2 \to \mathbb{R}^2$

$$\underline{\underline{A}} = \left[T(\underline{e}_1) \ T(\underline{e}_2) \right]$$

$$T(\underline{e}_1) = T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix}$$

$$T(\underline{e}_2) = T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix}$$

$$\Rightarrow \underline{\underline{A}} = \begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}$$

eg: $T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$

Find $\underline{\underline{A}}$.

$T\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$

Let $\underline{v}_1 = \begin{pmatrix}1\\1\end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix}1\\-1\end{pmatrix}$

Want $\underline{e}_1 = c_1\underline{v}_1 + c_2\underline{v}_2$

$\underline{e}_2 = d_1\underline{v}_1 + d_2\underline{v}_2$

We want $\begin{pmatrix}\underline{v}_1 & \underline{v}_2\end{pmatrix}\begin{pmatrix}c_1 & d_1\\ c_2 & d_2\end{pmatrix}$

$$= \begin{pmatrix}c_1\underline{v}_1 & d_1\underline{v}_1\\ +c_2\underline{v}_2 & +d_2\underline{v}_2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}\underline{e}_1 & \underline{e}_2\end{pmatrix}$$

so this is just $\begin{pmatrix}\underline{v}_1 & \underline{v}_2\end{pmatrix}^{-1}$

$$\begin{pmatrix} \underline{v}_1 & \underline{v}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} \underline{v}_1 & \underline{v}_2 \end{pmatrix}^{-1} = \frac{1}{-2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

Now $T(\underline{e}_1) = T(c_1 \underline{v}_1 + c_2 \underline{v}_2)$

$$= c_1 T(\underline{v}_1) + c_2 T(\underline{v}_2)$$

$$= \frac{1}{2}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Similarly $T(\underline{e}_2) = T(d_1 \underline{v}_1 + d_2 \underline{v}_2)$

$$= d_1 T(\underline{v}_1) + d_2 T(\underline{v}_2)$$

$$= \frac{1}{2}\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \underline{\underline{A}} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Shortcut:

Notice above this means we could've just computed:

$$\underline{\underline{A}} = \begin{bmatrix} T(\underline{e}_1) & T(\underline{e}_2) \end{bmatrix} = \begin{bmatrix} T(\underline{v}_1) & T(\underline{v}_2) \end{bmatrix} \begin{bmatrix} \underline{v}_1 & \underline{v}_2 \end{bmatrix}^{-1}$$

$$= \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

which is the same as before.