Welcome to Math 54!

Office hours: 1064 Evans, Tue 8:30-10AM

- Feel free to go to any office hours of any GSI

Linear System of Equations

e.g.

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ 3x_1 + 4x_2 &= 5 \end{aligned}$$ (equation form)

$$\rightarrow \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$ (matrix-vector form)

$$\to \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 3 & 4 & 5 \end{array}\right)$$ (augmented matrix form)

Matrices / vectors

Matrix: box of numbers, e.g. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$

size of matrix: # rows $\times$ # columns

$$\to \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} : 3 \times 2 \text{ matrix}$$

(3-by-2) matrix

Vector: Column of numbers, e.g. $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$

$\rightarrow$ or a 4×1 matrix

Operations (easy ones):

Addition: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 7 & 9 \end{pmatrix}$

$\boxed{\text{entry-wise}}$

Scalar multiplication: $5 \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 5 & 10 \\ 15 & 20 \end{pmatrix}$

Matrix-vector multiplication

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ? \\ ? \end{pmatrix}$$

2×2 matrix $\quad 2 \times 1$ vector $\quad\quad 2 \times 1$ vector

only possible if these numbers match

→ outer numbers become size of output

How?

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1x_1 + 2x_2 \\ ? \end{pmatrix}$$

- Go across matrix, down the vector, multiply and add

$$\rightarrow \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1x_1 + 2x_2 \\ 3x_1 + 4x_2 \end{pmatrix}$$

<u>This is why:</u>

$$\begin{matrix} 1x_1 + 2x_2 = 3 \\ 3x_1 + 4x_2 = 5 \end{matrix} \quad \longleftrightarrow \quad \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

How to solve? Row reduction

"also known as Gaussian Elimination"

1) Convert to augmented matrix

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ 3x_1 + 4x_2 &= 5 \end{aligned} \quad \longleftrightarrow \quad \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 3 & 4 & 5 \end{array}\right)$$

this is called diagonal of matrix

2) Add/subtract, multiply rows by scalars to make everything below diagonal zero

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ 3x_1 + 4x_2 &= 5 \end{aligned} \longleftrightarrow \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 3 & 4 & 5 \end{array}\right)$$

$$\boxed{\text{row } 2 - 3 \times \text{row } 1}$$

$$\Rightarrow \begin{aligned} x_1 + 2x_2 &= 3 \\ 0x_1 - 2x_2 &= -4 \end{aligned} \longleftrightarrow \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -2 & -4 \end{array}\right)$$

- now below diagonal is zero

$\longrightarrow$ called row echelon form

3 Make diagonal $=1$, everything not on diagonal zero

$$
\begin{aligned}
x_1 + 2x_2 &= 3 \\
0x_1 - 2x_2 &= -4
\end{aligned}
\quad \longleftrightarrow \quad
\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -2 & -4 \end{array}\right)
$$

$$\boxed{\text{row } 2 \times -\tfrac{1}{2}}$$

$$
\Rightarrow \quad
\begin{aligned}
x_1 + 2x_2 &= 3 \\
0x_1 + x_2 &= 2
\end{aligned}
\quad \longleftrightarrow \quad
\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & -2 \end{array}\right)
$$

$$\boxed{\text{row } 1 - 2 \times \text{row } 2}$$

$$
\Rightarrow \quad
\begin{aligned}
x_1 + 0x_2 &= -1 \\
0x_1 + x_2 &= 2
\end{aligned}
\quad \longleftrightarrow \quad
\left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right)
$$

→ And we are done!

Notes: - When row reducing, can add any rows to any other rows

---

Pivots: We saw above:

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 3 & 4 & 5 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -2 & -4 \end{array}\right)$$

In row echelon form,

\# non-zero entries on diagonal is called \# pivots.

1) If \# pivots = \# rows, solution always exists

e.g. $\begin{aligned} x_1 &= -1 \\ x_2 &= 2 \end{aligned} \longleftrightarrow \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right)$

2) If # pivots < # rows, solution might exist

e.g. $x_1 = -1$
$0 = 1$ $\longleftrightarrow$ $\left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 0 & 1 \end{array}\right)$

✗ nonsense

→ no solution

e.g. $x_1 = -1$
$0 = 0$ $\left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 0 & 0 \end{array}\right)$

ok

→ solution exists

⇒ sometimes ok, depends on right hand side.