Midterm 2 Review B **Name:** Solutions (Brief)
Date: 23/10/2022
Math 54: Fall 2022

Problem 1

True/False. Justify your answers.

1. If A is row equivalent to B they have the same eigenvalues. False
2. If A is non-invertible, it has at least one zero eigenvalue. True
3. Let $T : V \to V$ be a linear transformation and $\mathcal{B}, \mathcal{C}$ be two bases for V. Then $A_{B,B}$ and $A_{C,C}$ have the same eigenvalues. True

Problem 2

Let

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$ $, \underline{\underline{A}} = \underline{\underline{P}}\,\underline{\underline{D}}\,\underline{\underline{P}}^{-1}, \underline{\underline{A}}^5 = \underline{\underline{P}}\,\underline{\underline{D}}^5\,\underline{\underline{P}}^{-1}$

Compute A^5.

evals: $|\underline{\underline{A}} - \lambda \underline{\underline{I}}| = \begin{vmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)(\lambda^2 - 2\lambda) = 0$

$\lambda = 0, 1, 2$

evecs: $\lambda = 0$, $\underline{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ $\quad \lambda = 1$, $\underline{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\quad \lambda = 2$, $\underline{v} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$\to \underline{\underline{P}} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \underline{\underline{D}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$\to \underline{\underline{D}}^5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 32 \end{pmatrix}, \underline{\underline{P}}^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

$\to \underline{\underline{P}}\,\underline{\underline{D}}^5\,\underline{\underline{P}}^{-1} = \begin{pmatrix} 16 & 0 & -16 \\ 0 & 1 & 0 \\ -16 & 0 & 16 \end{pmatrix}$

Problem 3

True/False. Justify your answers.

1. If λ is an eigenvalues of A, it has geometric multiplicity at least 1. T
2. If an $n \times n$ matrix A has an eigenvalue λ with geometric multiplicity n, then A is a diagonal matrix. T
3. Any eigenvector of a matrix A is in the column space of A. F
4. If a square matrix A is diagonalisable and invertible, then A^{-1} is also diagonalisable. T

Problem 4

Suppose V and W are vector spaces and $T : V \to W$ is an invertible linear map. Suppose $\mathcal{B} = \{\mathbf{v}_1, ..., \mathbf{v}_n\}$ is a basis of V. Show that $\mathcal{S} = \underbrace{\{T(\mathbf{v}_1), ..., T(\mathbf{v}_n)\}}_{\text{Show lin. independent.}}$ is a basis of W.

$\alpha_1 T(\underline{v}_1) + \dots + \alpha_n T(\underline{v}_n) = 0$

$\to T(\alpha_1 \underline{v}_1) + \dots + T(\alpha_n \underline{v}_n) = 0, \quad T(\underbrace{\alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n}_{=\underline{0}}) = 0$

$\Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_n = 0$, as V a basis.

Problem 5

Find a basis for the $\text{Col}(A)^\perp, \text{Col}(B)^\perp$

$\text{Col}(A)^\perp = \text{Null}(\underline{A}^T)$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}$$

$\underline{A}^T = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$

$\underline{B}^T = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -6 \end{pmatrix}$

$\text{Null}\, \underline{A}^T = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\text{Null}(\underline{B}^T) = \text{Span}\left\{ \begin{pmatrix} -2 \\ 6 \\ 1 \end{pmatrix} \right\}$

Problem 6

Let W be a subspace of $\mathbb{R}^n$. Show that

$$W^\perp = \{\mathbf{u} \in \mathbb{R}^n \text{ such that } \mathbf{u} \cdot \mathbf{w} = 0 \text{ for all } \mathbf{w} \in W\}$$

is a subspace of $\mathbb{R}^n$.

1) $\underline{0} \in W^\perp, \quad \underline{0} \cdot \underline{w} = 0$

2) $\underline{v}_1, \underline{v}_2 \in W^\perp, \quad (\underline{v}_1 + \underline{v}_2) \cdot \underline{w} = \underline{v}_1 \cdot \underline{w} + \underline{v}_2 \cdot \underline{w} = 0$

$\to \underline{v}_1 + \underline{v}_2 \in W^\perp$

3) $\underline{v} \in W^\perp, \quad c\underline{v} \cdot w = c(\underline{v} \cdot \underline{w}) = 0$

$\to c\underline{v} \in W^\perp$

Problem 7

True/False. Justify your answers.

1. $\mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} = 0$ T
2. If $\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \|\mathbf{u} + \mathbf{v}\|^2$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. T
3. For an $m \times n$ matrix A, vectors in the null space of A are orthogonal to vectors in the row space of A. T

Problem 8

Determine if the sets of vectors is orthonormal. If it is only orthogonal, normalise the vectors to produce an orthonormal set.

$$\underset{\uparrow \atop \underline{v}_1}{\begin{bmatrix} \sqrt{2} \\ 3 \\ 3 \end{bmatrix}}, \quad \underset{\uparrow \atop \underline{v}_2}{\begin{bmatrix} 6 \\ -\sqrt{2} \\ -\sqrt{2} \end{bmatrix}}, \quad \underset{\uparrow \atop \underline{v}_3}{\begin{bmatrix} 0 \\ -\sqrt{10} \\ \sqrt{10} \end{bmatrix}}$$

$\underline{v}_1 \cdot \underline{v}_2 = 0$

$\underline{v}_1 \cdot \underline{v}_3 = 0$, orthogonal

$\underline{v}_2 \cdot \underline{v}_3 = 0$

$\|\underline{v}_1\| = \sqrt{2+9+9} = \sqrt{20}$

$\|\underline{v}_2\| = \sqrt{36+2+2} = \sqrt{40}$

$\|\underline{v}_3\| = \sqrt{10+10} = \sqrt{20}$

$\rightarrow \left\{ \frac{1}{\sqrt{20}}\begin{pmatrix} \sqrt{2} \\ 3 \\ 3 \end{pmatrix}, \frac{1}{\sqrt{40}}\begin{pmatrix} 6 \\ -\sqrt{2} \\ -\sqrt{2} \end{pmatrix}, \frac{1}{\sqrt{20}}\begin{pmatrix} 0 \\ -\sqrt{10} \\ \sqrt{10} \end{pmatrix} \right\}$

Problem 9

Determine whether each of these sets are an orthogonal basis for $\mathbb{R}^3$

$$\underbrace{\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}}_{\text{Not orthogonal}}, \underbrace{\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}}_{\text{Orthogonal}}$$

Problem 10

Find the distance of the point $\mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ from the two dimensional subspace $W \subset \mathbb{R}^3$ spanned by

$\underbrace{\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}$ set is orthogonal

$$\Rightarrow \operatorname{proj}_{u_1} x + \operatorname{proj}_{u_2} x = \frac{\begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \frac{\begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$= \frac{-2}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \frac{6}{3} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

Problem 11

True/False. Justify your answers.

1. Not every orthogonal set in $\mathbb{R}^n$ is linearly independent. T
2. If the columns of an $m \times n$ matrix A are orthonormal, then $\|\mathbf{x}\| = \|A\mathbf{x}\|$. T
3. The orthogonal projection of $\mathbf{y}$ onto $\mathbf{v}$ is the same as the orthogonal projection of $\mathbf{y}$ onto $c\mathbf{v}$ whenever $c \neq 0$. T
4. A matrix with orthonormal columns is invertible. F

Problem 12

Find an orthogonal basis for the null space of

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{pmatrix} \longrightarrow \begin{aligned} x_1 \quad - x_3 - 2x_4 &= 0 \\ x_2 + 2x_3 + 3x_4 &= 0 \end{aligned}$$

$$x_3 = x_3 \qquad x_4 = x_4$$

$$\longrightarrow \begin{aligned} x_1 &= x_3 + 2x_4 \\ x_2 &= -2x_3 - 3x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned} \rightarrow x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{Basis null: } \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\} \xrightarrow{\text{Gram-Schmidt}} \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{4}{3} \\ 1 \end{pmatrix} \right\}$$