**Midterm 2 Review A** **Name:** Solutions (Brief)
**Date: 23/10/2022**
**Math 54: Fall 2022**

## Problem 1

What conditions must a set $A \subset B$ satisfy to be a subspace of $B$?

1) $0 \in A$
2) Closure addition
3) Closure scalar multiply

$0 \in A$, $a_1 + a_2 \in A$ if $a_1, a_2 \in A$, $ca \in A$ if $a \in A$

## Problem 2

Consider a linear operator $T : V \to W$ is a subspace. Is the null space of $T$ a subset of $V$ or of $W$? Show that it is a subspace of $V$ or $W$.

Subset of V

$T(0) = 0$, $v_1, v_2 \in \text{Null } T$: $T(v_1 + v_2) = T(v_1) + T(v_2) = 0$, $v \in \text{Null } T$: $T(cv) = cT(v) = 0$

## Problem 3

For each of the following, show that the set of vectors is a basis and find the coordinate of the following vectors in their respective bases:

1. $\mathbf{v} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$, $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\begin{pmatrix} 1 & 0 & 1 \\ 1 & -2 & 0 \\ -1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & 2 \end{pmatrix}$ → pivot each col.
– linearly independent
→ basis

$[\underline{v}]_{\mathcal{B}} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$

2. $\mathbf{v} = 1 + x + x^2$, $\mathcal{B} = \left\{ \underbrace{1, 1 - 2x, x - 2x^2}_{\text{lin. independent, so basis}} \right\}$

$[\underline{v}]_{\mathcal{B}} = \begin{pmatrix} \frac{7}{4} \\ -\frac{3}{4} \\ -\frac{1}{2} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 7 \\ -3 \\ -2 \end{pmatrix}$

## Problem 4

Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ and rank of T be 2. What is the nullity of $T$?

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## Problem 5

Find the change of basis matrix $P_{C \leftarrow B}$

$$\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix} \right\}, \mathcal{C} = \left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} \right\}$$

$$P_{C \leftarrow B} = \underline{P}_C^{-1} \underline{P}_B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

## Problem 6

Find the matrix $A_{B,C}$ of the linear transformation $T$ relative to the bases $\mathcal{B}, \mathcal{C}$. Is this transformation one-to-one or onto?

$$T\left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 + 2x_2 \\ 5x_1 - x_2 \\ 6x_1 + 3x_2 \end{bmatrix}, \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}, \mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$\underline{A}_{B,C} = \left( [T(b_1)]_C \; [T(b_2)]_C \right) = \left( \left[\begin{pmatrix} 5 \\ 3 \\ 12 \end{pmatrix}\right]_C \; \left[\begin{pmatrix} 3 \\ 4 \\ 9 \end{pmatrix}\right]_C \right)$$

$$= \begin{pmatrix} -15 & -17 \\ 6 & 10 \\ 17 & 16 \end{pmatrix}$$

## Problem 7

Consider the map $T : \mathbb{P}^3 \to \mathbb{P}^2$ where $T(f) = f'$.

1. Find the matrix of $T$ with respect to the standard bases. $\mathcal{E} = \{1, x, x^2, x^3\}$
2. Find bases for $\mathbb{P}^3, \mathbb{P}^2$ such that the matrix for $T$ relative to these bases is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

3. Is this map one-to-one or onto?

1) $T(1) = 0$, $T(x) = 1$, $T(x^2) = 2x$, $T(x^3) = 3x^2$, $[T]_{\mathcal{E}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

2) $\mathcal{B} = \{ x, \frac{1}{2}x^2, \frac{1}{3}x^3, 1 \}$

c) Onto (Pivot each row), Not 1-to-1 (No pivot in each col)

**Problem 8**

Find the determinants of the following matrices

$$A = \begin{bmatrix} 2 & -4 \\ 5 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 1 & -4 & -2 \\ 0 & 5 & -1 \\ 0 & 0 & 7 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 2 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

18 35 2

**Problem 9**

Let $T : V \to V$ be a linear transformation. What does it mean for $\mathbf{v}$ to be an eigenvector of $T$?

$T(\underline{v}) = \lambda \underline{v}$

**Problem 10**

Consider $T : \mathbb{P}^2 \to \mathbb{P}^2$ defined by $T(f) = f(0) + f' \cdot x + f(0) \cdot x^2$

1. Is $f(x) = 1$ an eigenvector of this transformation? If so what is the corresponding eigenvalue?
2. Is $f(x) = x^2$ an eigenvector of this transformation? If so what is the corresponding eigenvalue?

1) $T(f(x)=1) = 1 + 0 \cdot x + 1 \cdot x^2 = 1 + x^2 \neq \lambda \cdot 1$

– Not evec

2) $T(f(x)=x^2) = 0 + 2x \cdot x + 0 \cdot x^2$
$= 2x^2$ – Is evec w/ eval $= 2$

**Problem 11**

What is the characteristic polynomial for a square matrix $A$?

$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}})$

### Problem 12

Find the characteristic polynomial and all eigenvalues of the following matrix. Is the matrix diagonalisable? If so diagonalise it.

$$\underline{A} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

For evals: $\det(\underline{A} - \lambda \underline{I}) = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$\to (1-\lambda)(\lambda^2 - 3\lambda) = 0$

(Distinct evals → diagonalisable) $\to \lambda = 0, 1, 3$

For evecs: $\lambda = 0$, $\underline{v} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$, $\lambda = 1$, $\underline{v} = \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \\ 1 \end{pmatrix}$, $\lambda = 3$, $\underline{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

### Problem 13

Compute the following given that $\underline{A} = \underline{P}\underline{D}\underline{P}^{-1}$, $\underline{P} = \begin{pmatrix} -2 & \frac{1}{2} & 1 \\ 1 & -3/2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $\underline{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$$\mathbf{u} = \begin{bmatrix} 3 \\ -4 \\ -5 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$$

1. $\mathbf{u} \cdot \mathbf{v}$ $= 18 + 8 - 15 = 11$
2. $\text{proj}_{\mathbf{u}} \mathbf{u}$ $= \underline{u}$
3. $\text{proj}_{\mathbf{u}} \mathbf{v}$ $= \frac{\underline{u} \cdot \underline{v}}{\underline{u} \cdot \underline{u}} \underline{u} = \frac{11}{9+16+25} \begin{pmatrix} 3 \\ -4 \\ -5 \end{pmatrix} = \frac{11}{50} \begin{pmatrix} 3 \\ -4 \\ -5 \end{pmatrix}$
4. $\text{proj}_{\mathbf{v}} \mathbf{v}$ $= \underline{v}$

### Problem 14

Find the distance between the two vectors

$$\mathbf{u} = \begin{bmatrix} 3 \\ -4 \\ -5 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix} \to \|\underline{u} - \underline{v}\|$$

$$= \left\| \begin{matrix} -3 \\ -2 \\ -8 \end{matrix} \right\| = \sqrt{9 + 4 + 64} = \sqrt{77}$$

### Problem 15

Use the Gram-Schmidt process to find a set of orthonormal vectors that have the same span as

$$\left\{ \underset{\underline{v}_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}, \underset{\underline{v}_2}{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}}, \underset{\underline{v}_3}{\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}} \right\}.$$

$\underline{w}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\underline{w}_2 = \underline{v}_2 - \text{proj}_{\underline{w}_1} \underline{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} - \frac{\underline{w}_1 \cdot \underline{v}_2}{\underline{w}_1 \cdot \underline{w}_1} \underline{w}_1$

$= \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}$

$\underline{w}_3 = \underline{v}_3 - \underbrace{\text{proj}_{\underline{w}_1} \underline{v}_3}_{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{\underline{v}_3 \cdot \underline{w}_1}{\underline{w}_1 \cdot \underline{w}_1} \underline{w}_1} - \underbrace{\text{proj}_{\underline{w}_2} \underline{v}_3}_{\frac{\underline{v}_3 \cdot \underline{w}_2}{\underline{w}_2 \cdot \underline{w}_2} \underline{w}_2 = \frac{1}{5}\begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}} = \begin{pmatrix} 0 \\ 3/5 \\ -1 \\ -6/5 \end{pmatrix}$

$\to \{\underline{w}_1, \underline{w}_2, \underline{w}_3\}$