Midterm 1 Review A **Name:** ____________________
Date: 19/9/2022
Math 54: Fall 2022

Problem 1

What are the properties of a linear transformation $T : X \to Y$?

Problem 2

Consider the following matrices

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

1. What are the domain and codomains for the corresponding linear transformations T_A, T_B?
2. Are these linear transformations one-to-one, onto, or a bijection?

Problem 3

1. Define the range and kernel of a linear transformation.
2. Find the range and kernel of the linear transformation T_A given by the matrix $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix}$.

Problem 4

1. Give an example of a basis of $\mathbb{R}^3$. What conditions must a set of vectors have in order to be a basis?
2. Define the dimension of subspace W.

Problem 5

Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ and rank of T be 2. What is the nullity of T?

The problems from this point are slightly more tricky. They will test your understanding.

Problem 6

Let $T : \mathbb{R}^n \to \mathbb{R}^m$. Let A be the associated matrix.

1. What is the size of A?
2. Write whether each of the following statements means that T is one-to-one, onto, or bijective, or none.

   (a) For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, if $\mathbf{x} \neq \mathbf{y}$ then $T(\mathbf{x}) \neq T(\mathbf{y})$.
   (b) For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, if $\mathbf{x} = \mathbf{y}$ then $T(\mathbf{x}) = T(\mathbf{y})$.
   (c) For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, if $T(\mathbf{x}) = T(\mathbf{y})$ then $\mathbf{x} = \mathbf{y}$.
   (d) For every $\mathbf{y} \in \mathbb{R}^m$, there exists $\mathbf{x} \in \mathbb{R}^n$ such that $T(\mathbf{x}) = \mathbf{y}$.
   (e) A row reduces to have a pivot in every column.
   (f) A row reduces to have a pivot in every row.
   (g) $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$.
   (h) Any solution to $A\mathbf{x} = \mathbf{b}$ is unique.
   (i) The columns of A are linearly independent.
   (j) The columns of A span $\mathbb{R}^m$.

Problem 6

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Explain why every vector in Span$(\mathbf{u}, \mathbf{v})$ is also in Span$(\mathbf{u}, \mathbf{v} - \mathbf{u})$.

Problem 7

Label each of the following true/false.

1. If A is an $m \times n$ matrix whose columns do not span $\mathbb{R}^m$, then the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent for some $\mathbf{b} \in \mathbb{R}^m$.
2. If S is a linearly dependent set, then each vector is a linear combination of the other vectors in S.
3. If $\mathbf{x}$ and $\mathbf{y}$ are linearly independent, and $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly dependent, then $\mathbf{z}$ is in Span$\{\mathbf{x}, \mathbf{y}\}$.
4. If A and B are invertible $n \times n$ invertible matrices, then AB is invertible and $(AB)^{-1} = A^{-1}B^{-1}$

Problem 8

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a linearly dependent set in $\mathbb{R}^3$. Explain why $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ must be linearly dependent as well.

Problem 9

For each of the following, determine if W is a subspace of V :

1. V = $\mathbb{R}^2$, and W is the line in $\mathbb{R}^2$ given by the equation $y = 2x$.
2. V = $\mathbb{R}^2$, and W is the circle in $\mathbb{R}^2$ given by the equation $x^2 + y^2 = 1$.
3. V = $\mathbb{R}^2$, and W is the set in $\mathbb{R}^2$ given by the equation $x^2 + y^2 = 0$.
4. V is the space of 3×3 matrices, and W is the subset of invertible 3×3 matrices.
5. V is the space of polynomials, and W is the set of polynomials of degree equal to 3.
6. V is the space of polynomials, and W is the set of polynomials of degree less or equal to 3.

Problem 10

Find a matrix A such that Range(A) $= \left\{ \begin{bmatrix} a-b \\ b \\ 3a+b \end{bmatrix}, \ a, b \in \mathbb{R} \right\}$.

Problem 11

Let $T : V \to W$ be a linear transformation. Show that Range(T) is a subspace of W.

Problem 12

Let $M_{2\times2}$ be the space of 2×2 matrices. Define $T : M_{2\times2} \to M_{2\times2}$ by $T(A) = A - A^T$.

1. Show that T is linear.
2. Determine the kernel and range of T.