Quiz # 8 **Name:** Solutions
Date: 3/11/2022
Math 54: Fall 2022

Consider the vector space of real valued functions defined on the domain $[-1,1]$ equipped the standard inner product.

Problem 1 - 3 Points

Find an orthogonal set that has the same span as $\{1, x, x^2\}$ on this domain.

$u_1 = 1$

$u_2 = x - \text{proj}_1 x \quad , \quad \text{proj}_1 x = \dfrac{\int_{-1}^{1} x \, dx}{\int_{-1}^{1} 1 \, dx} 1 = 0$ odd fn.

$= x$

$u_3 = x^2 - \text{proj}_1 x^2 - \text{proj}_x x^2 \quad , \quad \text{proj}_x x^2 = \dfrac{\int_{-1}^{1} x^3 \, dx}{\int_{-1}^{1} x^2 \, dx} x = 0$ odd

$= x^2 - \frac{1}{3}$

$\text{proj}_1 x^2 = \dfrac{\int_{-1}^{1} x^2}{\int_{-1}^{1} 1} 1 = \dfrac{\frac{2}{3}}{2} = \frac{1}{3}$

$\rightarrow \left\{ 1, x, x^2 - \frac{1}{3} \right\}$

Problem 2 - 3 Points

Find the norms of each function in the orthogonal set you found above.

$\|\underline{u}_1\|^2 = \int_{-1}^{1} 1 \, dx = 2 \quad , \quad \|\underline{u}_1\| = \sqrt{2}$

$\|\underline{u}_2\|^2 = \int_{-1}^{1} x^2 \, dx = \frac{2}{3} \quad , \quad \|\underline{u}_2\| = \sqrt{\frac{2}{3}}$

$\|\underline{u}_3\|^2 = \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 dx = \int_{-1}^{1} x^4 - \frac{2}{3}x^2 + \frac{1}{9}x \, dx$

$$= \frac{1}{5}x^5 - \frac{2}{9}x^3 + \frac{1}{9}x \Big|_{-1}^{1}$$

$$= \frac{2}{5} - \frac{4}{9} + \frac{2}{9} = \frac{8}{45}$$

$$\|u_3\| = \sqrt{\tfrac{8}{45}}$$

Problem 3 - 3 Points

Find the best approximation of the function e^x on $[-1, 1]$ using the functions $\{1, x\}$ via projection.

$$\rightarrow \quad \text{proj}_1 e^x + \text{proj}_x e^x = \frac{\int_{-1}^{1} e^x \, dx}{\int_{-1}^{1} 1 \, dx} 1 + \frac{\int_{-1}^{1} x e^x \, dx}{\int_{-1}^{1} x^2 \, dx} x$$

$$= \frac{e - \frac{1}{e}}{2} + \underbrace{\frac{xe^x \big|_{-1}^{1} - \int_{-1}^{1} e^x \, dx}{2/3}}_{\frac{e + \frac{1}{e} - (e - \frac{1}{e})}{2/3} = \frac{\cancel{2}}{e} \cdot \frac{3}{\cancel{2}} = \frac{3}{e}} x$$

$$= \frac{e - \frac{1}{e}}{2} + \frac{3}{e} x$$

Problem 4 - 1 Points

[redacted]

They're all real places.

— I actually grew up pretty close to Woolloomooloo