Quiz # 7 **Name:** Solutions
Date: 20/10/2022
Math 54: Fall 2022

Problem 1 - 5 Points

Consider the linear transformation $T : \mathbb{P}_2 \to \mathbb{P}_2$ given by $T\big(p(x)\big) = p(0) + p(1) \cdot x + p'(x) \cdot 2x$.

1. Using the standard basis $\mathcal{E} = \{1, x, x^2\}$, find the standard matrix of the transformation $[T]_{\mathcal{E}}$.
2. Find a basis $\mathcal{B}$ such that the matrix representation of the transformation under this basis $[T]_{\mathcal{B}}$ is diagonal. What is $[T]_{\mathcal{B}}$ under this basis?

1) $T(1) = 1 + x$

$T(x) = 0 + x + 2x = 3x$

$T(x^2) = 0 + x + (2x)(2x) = x + 4x^2$

$\to [T]_{\mathcal{E}} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & 1 \\ 0 & 0 & 4 \end{pmatrix}$

2) Eigenvalues : $\det \begin{pmatrix} 1-\lambda & 0 & 0 \\ 1 & 3-\lambda & 1 \\ 0 & 0 & 4-\lambda \end{pmatrix} = (1-\lambda) \begin{vmatrix} 3-\lambda & 1 \\ 0 & 4-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda = 1, 3, 4$

Eigenvectors: $\lambda = 1$, null $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $\underline{v}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

$\lambda = 3$, null $\begin{pmatrix} -2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $\underline{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\lambda = 4$, null $\begin{pmatrix} -3 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$, $\underline{v}_4 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$\mathcal{B} = \{-2+x, x, x-x^2\}$, $[T]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

Problem 2 - 5 Points

1. Show that $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.
2. If $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$, what can you say about $\mathbf{u}, \mathbf{v}$? (Hint: This is called the triangle inequality. Why? Draw a picture to see what's going on!)

1) $$\|\underline{u}+\underline{v}\|^2 = (\underline{u}+\underline{v}) \cdot (\underline{u}+\underline{v})$$

$$= \underline{u}\cdot\underline{u} + 2\underline{u}\cdot\underline{v} + \underline{v}\cdot\underline{v}$$

$$= \|\underline{u}\|^2 + \|\underline{v}\|^2 + 2\|\underline{u}\|\,\|\underline{v}\| \cos\theta$$

$$(\|\underline{u}\| + \|\underline{v}\|)^2 = \|\underline{u}\|^2 + \|\underline{v}\|^2 + 2\|\underline{u}\|\,\|\underline{v}\|$$

$$\|\underline{u}\|\,\|\underline{v}\| \geq \|\underline{u}\|\,\|\underline{v}\| \cos\theta, \text{ as } \cos\theta \in [-1,1]$$

So claim holds.

2) $\cos\theta = 1$ if equality, so $\underline{u}, \underline{v}$ parallel.