Problem 1a

Determine all values of k such that $\begin{bmatrix} 2 \\ 3 \\ k \end{bmatrix}$ is in the span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \right\}$.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 2 & 1 & 3 & 3 \\ 0 & 3 & 3 & k \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -3 & -3 & -1 \\ 0 & 3 & 3 & k \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -3 & -3 & -1 \\ 0 & 0 & 0 & k-1 \end{array}\right)$$

For system to be consistent, $k=1$.

Problem 1b

Is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^3$?

No. Since does not span $\mathbb{R}^3$.

e.g. $\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$ not in the span.

Also vectors are not linearly independent.

$$\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}.$$

Problem 2

Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation such that

$$T\left(\begin{bmatrix}1\\-3\\0\end{bmatrix}\right) = \begin{bmatrix}-2\\3\\-4\\0\end{bmatrix},\ T\left(\begin{bmatrix}0\\3\\0\end{bmatrix}\right) = \begin{bmatrix}3\\-3\\6\\0\end{bmatrix},T\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\3\\2\\0\end{bmatrix}\}$$

1. Write the standard matrix A of the transformation T.
2. Is $A\mathbf{x} = \mathbf{b}$ consistent for any $\mathbf{b} \in \mathbb{R}^4$?
3. Find a basis for the column and null space of this transformation.
4. Verify the rank nullity theorem for this transformation.

1) $T\begin{pmatrix}1\\0\\0\end{pmatrix} = T\left(\begin{pmatrix}1\\-3\\0\end{pmatrix} + \begin{pmatrix}0\\3\\0\end{pmatrix}\right) = \begin{pmatrix}-2\\3\\-4\\0\end{pmatrix} + \begin{pmatrix}3\\-3\\6\\0\end{pmatrix} = \begin{pmatrix}1\\0\\2\\0\end{pmatrix}$

$T\begin{pmatrix}0\\1\\0\end{pmatrix} = \frac{1}{3}T\begin{pmatrix}0\\3\\0\end{pmatrix} = \begin{pmatrix}1\\-1\\2\\0\end{pmatrix}$

$T\begin{pmatrix}0\\0\\1\end{pmatrix} = T\begin{pmatrix}0\\1\\1\end{pmatrix} - T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\3\\2\\0\end{pmatrix} - \begin{pmatrix}1\\-1\\2\\0\end{pmatrix} = \begin{pmatrix}0\\4\\0\\0\end{pmatrix}$

$\Rightarrow \underline{A} = \begin{pmatrix}1 & 1 & 0\\0 & -1 & 4\\2 & 2 & 0\\0 & 0 & 0\end{pmatrix}$

2) No, more columns than rows $\underline{A}$

$\rightarrow$ can't have pivot in each row, so T is not onto

3) $\underline{A} \sim \begin{pmatrix}1 & 1 & 0\\0 & -1 & 4\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix}$ $\Rightarrow$ Basis col. space $= \left\{\begin{pmatrix}1\\0\\2\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\2\\0\end{pmatrix}\right\}$

2 pivots

$\Rightarrow$ rank $\underline{A} = 2$

For null space: $\underline{A} \sim \begin{pmatrix}1 & 0 & 4\\0 & 1 & 4\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix} \rightarrow \begin{matrix}x_1 + 4x_3 = 0\\x_2 + 4x_3 = 0\end{matrix} \Rightarrow$ Basis null space $= \left\{\begin{pmatrix}-4\\-4\\1\end{pmatrix}\right\}$

nullity $= 1$

4) Rank $\underline{A}$ + Nullity $\underline{A} = 2+1 = 3 = \#$ columns of $\underline{A}$ ✓

Problem 3

For each of the following determine whether the statements are true or false. If true explain why. If false explain why not with a counterexample or otherwise.

1. Let A be invertible. Then A^T is invertible.

True.

Possible explanations:

1) $\operatorname{rank}(\underline{\underline{A}}) = \operatorname{rank}(\underline{\underline{A}}^T)$

2) $(\underline{\underline{A}}\,\underline{\underline{A}}^{-1})^T = (\underline{\underline{A}}^{-1})^T \underline{\underline{A}}^T = \underline{\underline{I}}$

$\rightarrow \underline{\underline{A}}^T$ invertible, $(\underline{\underline{A}}^T)^{-1} = (\underline{\underline{A}}^{-1})^T$

2. Let AB be invertible. Then both A and B are invertible.

False. e.g. $\underline{\underline{A}} = \begin{pmatrix} 1 & 0 \end{pmatrix}$, $\underline{\underline{B}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, neither invertible

but $\underline{\underline{A}}\,\underline{\underline{B}} = \begin{pmatrix} 1 \end{pmatrix}$ which is trivially invertible.

3. There exists a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T\left(\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $T\left(\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

False. If linear then $T\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = T\left(2\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}\right)$

$= 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

which is impossible.

4. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis of $\mathbb{R}^3$ and $T : \mathbb{R}^3 \to \mathbb{R}^3$ be some linear transformation. Then $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is a basis of $\mathbb{R}^3$.

False. Let $T(\underline{x}) = \underline{0}$. Then clearly

$T(\underline{v}_1) = T(\underline{v}_2) = T(\underline{v}_3) = \underline{0}$ not linearly independent.