

Last time Diagonalisation (of Matrices)

$$\underline{A} = \underline{P} \underline{D} \underline{P}^{-1}$$

\uparrow diagonal

- interpretation

- find a basis s.t. matrix representation of transformation is diagonal

In general: $T: X \rightarrow Y$, is there a basis B in which $[T]_B$ matrix is diagonal?

e.g.: $T: \mathbb{P}_2 \rightarrow \mathbb{P}_2$, $T(p)(x) = (x+1)p'(x)$

Let's look at standard basis, $\{1, x, x^2\} = \mathcal{E}$

$$[T]_{\mathcal{E}} = \begin{bmatrix} [T(1)]_{\mathcal{E}} & [T(x)]_{\mathcal{E}} & [T(x^2)]_{\mathcal{E}} \end{bmatrix}$$

$$= \begin{bmatrix} [0]_{\mathcal{E}} & [1+x]_{\mathcal{E}} & [2x+2x^2]_{\mathcal{E}} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix}, \text{ not diagonal.}$$

Try diagonalising: $\lambda=0, \lambda=1, \lambda=2$ (Upper triangular)

$\lambda=0$, eigenvector is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\lambda=2$ eigenvector is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\lambda=1$, eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, (try finding these yourself!)

→ Try basis $B = \{1, 1+x, 1+2x+x^2\}$

$$[T]_B = \left[[T(1)]_B \quad [T(1+x)]_B \quad [T(1+2x+x^2)]_B \right]$$

$$= \left[[0]_B \quad [1+x]_B \quad \underbrace{[(2x+2)(x+1)]_B}_{2x^2+4x+2} \right]$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \text{ diagonal!}$$

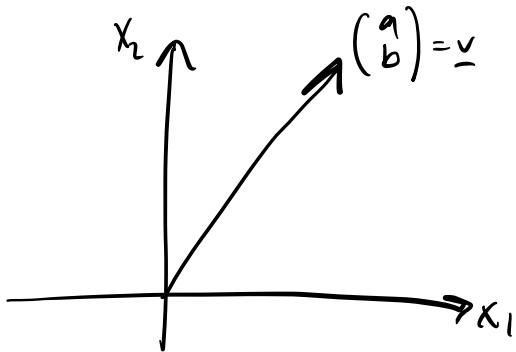
So in basis B , the matrix of T is diagonal!

Q) Do $[T]_E$ and $[T]_B$ have same eigenvalues?

A) Yes! Because they are similar to one another.

New Topic Inner Products (This is on Multivar 2)

- Basically how lengths / angles in \mathbb{R}^n work



What is length of \underline{v} ?

$$\|\underline{v}\| = \sqrt{a^2 + b^2}$$

- Pythagoras!

Def. Dot Product / inner Product

$$\underline{u}, \underline{v} \in \mathbb{R}^n, \text{ then } \underline{u} \cdot \underline{v} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

$$= u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

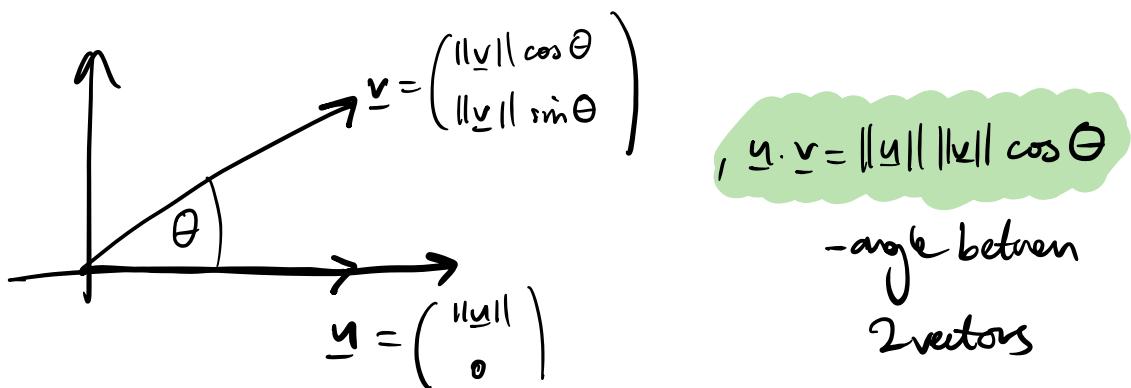
$$\text{e.g. } \underline{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \underline{v} = \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}, \underline{u} \cdot \underline{v} = 1 \cdot 0 + 2 \cdot 3 + 1 \cdot 4 = 10$$

$$\text{e.g. } \underline{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \underline{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \underline{u} \cdot \underline{v} \text{ not defined. Different size.}$$

$$\underline{\text{Def.}} \text{ Length of vector } \|\underline{v}\| = \sqrt{\underline{v} \cdot \underline{v}}, \underline{v} = \begin{pmatrix} a \\ b \end{pmatrix}$$

in \mathbb{R}^2 , thus what you are used to

Another way to think about this: (in \mathbb{R}^2)



Properties of Inner Product

$$1) \quad \underline{u} \cdot \underline{v} = \underline{v} \cdot \underline{u}$$

$$2) \quad (\underline{u} + \underline{v}) \cdot \underline{w} = \underline{u} \cdot \underline{w} + \underline{v} \cdot \underline{w}$$

$$3) \quad (c\underline{u}) \cdot \underline{v} = c(\underline{u} \cdot \underline{v})$$

$$4) \quad \underline{u} \cdot \underline{u} \geq 0, \text{ if } = 0, \text{ then } \underline{u} = 0$$

Def Unit vector, if $\|\underline{u}\|=1$

For $\underline{v} \in \mathbb{R}^n$, to normalise, just divide by norm $\|\underline{v}\|$,

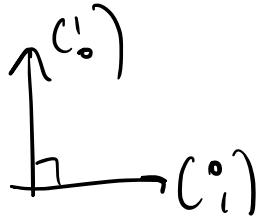
e.g. $\underline{v} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}, \|\underline{v}\| = \sqrt{16+1+4} = \sqrt{21}, \frac{\underline{v}}{\|\underline{v}\|} = \frac{1}{\sqrt{21}} \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$

Def. Orthogonal Vectors

Two vectors $\underline{u}, \underline{v}$ are orthogonal if $\underline{u} \cdot \underline{v} = 0$

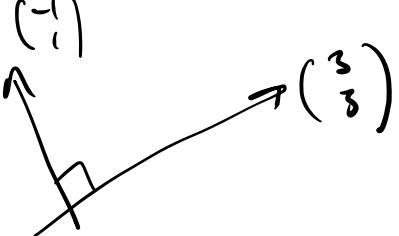
→ Interpretation, they are at right angle to each other

e.g.



$$(1, 0) \cdot (0, 1) = 0$$

e.g.



$$(-1, 1) \cdot (3/3, 3/3) = 3 - 3 = 0$$

Why? $\underline{u} \cdot \underline{v} = \|\underline{u}\| \cdot \|\underline{v}\| \cos \theta = 0$, $\underline{u}, \underline{v} \neq 0$

$\Leftrightarrow \cos \theta = 0$, as $\|\underline{u}\| \neq 0, \|\underline{v}\| \neq 0$

$\Leftrightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$ (i.e. rightangle)