

## Laol The

## Eigenvalue / Eigenvector

$$\underline{A}\underline{v} = \lambda\underline{v}, \quad \lambda \text{ eigenvalue, } \underline{v} \text{ eigenvector}$$

1)  $\det(\underline{A} - \lambda\underline{I})$ , characteristic polynomial

2) Solve  $\det(\underline{A} - \lambda\underline{I}) = 0 \rightarrow$  eigenvalues

3) Find null  $(\underline{A} - \lambda\underline{I})$  for eigenvectors

## Theorem

## Diagonalization

eg.  $\underline{A} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \underline{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$\underline{A}\underline{v}_1 = \begin{pmatrix} 3 \\ 3 \end{pmatrix},$$

- so eigenvector of  
eigenvalue 3

$$\underline{A}\underline{v}_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

- so eigenvector of  
eigenvalue -1

Let  $\underline{P} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ , compute  $\underline{A}\underline{P} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 3 & -1 \end{pmatrix}$

Let  $\underline{D} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$ , compute  $\underline{P}\underline{D} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 3 & -1 \end{pmatrix}$

$$\Rightarrow \underline{A}\underline{P} = \underline{P}\underline{D}, \quad \underline{A} = \underline{P}\underline{D}\underline{P}^{-1}$$

How to diagonalise:

1) Find eigenvalues  $\lambda_1, \lambda_2, \dots$

2) Find eigenvectors  $v_1, v_2, \dots$

3)  $\underline{P} = (v_1 \ v_2 \ \dots \ v_n)$ ,  $\underline{D} = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \dots & \lambda_n \end{pmatrix}$

then  $\underline{A} = \underline{P} \underline{D} \underline{P}^{-1}$

Does this always work? No.

eg.  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ , eigenvalues are 1, 1  
ie. 1 with multiplicity 2.

For eigenvectors:  $\text{null} \left[ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right]$

$= \text{null} \left[ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right] = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

→ Only 1 vector in basis for eigenvectors

→ Can't form  $\underline{P} = (v_1 \ v_2)$

- need 2 vectors.

Definition For a given eigenvalue  $\lambda'$  of  $\underline{\underline{A}}$ :

1) If char. poly.  $\det(\underline{\underline{A}} - \lambda' \underline{\underline{I}}) = (\lambda - \lambda')^k$  (stuff)

where  $k$  is the maximum power of  $(\lambda - \lambda')$ ,

$k =$  algebraic multiplicity of  $\lambda'$

2)  $\dim \text{null}(\underline{\underline{A}} - \lambda' \underline{\underline{I}}) =$  geometric multiplicity of  $\lambda'$

3)  $\text{null}(\underline{\underline{A}} - \lambda' \underline{\underline{I}})$  is called the eigenspace of  $\lambda'$

Theorem  $\underline{\underline{A}}$  is diagonalizable iff the algebraic multiplicity of every eigenvalue equals the geometric multiplicity.

Rule

$$1 \leq \text{Geometric multiplicity} \leq \text{Algebraic multiplicity}$$

FACT

A is  $n \times n$

the number of eigenvalues of A counting multiplicity  
sums to  $n$ .  
(ALGEBRAIC)

eg.  $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ ,  $\lambda=1$ ,  $\lambda=2$

eg.  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ,  $\lambda=1$ , multiplicity 3

eg.  $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ ,  $\lambda=2$ , multiplicity 2 (algebraic)  
- geometric multiplicity 1.

Theorem A is diagonalizable if and only if it has  $n$  linearly independent eigenvectors.

Theorem For each distinct eigenvalue of A, there exists at least 1 vector in the basis of the eigenspace.

→ If A has  $n$  distinct eigenvalues, it is diagonalizable.

eg: A =  $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ , is it diagonalizable?

$$\det(\underline{A} - \lambda \underline{I}) = \det \begin{pmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{pmatrix} = (1-\lambda)(4-\lambda) - 6 = 0$$

$$\rightarrow \lambda^2 - 5\lambda - 2 = 0, \lambda = \frac{5 \pm \sqrt{25+8}}{2}$$

→ 2 distinct eigenvalues. Yes diagonalizable.

eg. Is  $\underline{A} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 3 \end{pmatrix}$  diagonalizable?

$$\det(\underline{A} - \lambda \underline{I}) = \det \begin{pmatrix} 1-\lambda & -1 & 1 \\ -1 & 1-\lambda & 1 \\ -1 & -1 & 3-\lambda \end{pmatrix} = 0$$

$$\rightarrow (1-\lambda) \begin{vmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix} + 1 \begin{vmatrix} -1 & 1 \\ -1 & 3-\lambda \end{vmatrix} + 1 \begin{vmatrix} -1 & 1-\lambda \\ -1 & -1 \end{vmatrix} = 0$$

$$(1-\lambda) \left( (1-\lambda)(3-\lambda) + 1 \right) - (3-\lambda) + 1 + 1 + (1-\lambda) = 0$$

$$(1-\lambda)^2 (3-\lambda) + 1 - \cancel{3} + \cancel{\lambda} + 1 + 1 - \cancel{1} - \lambda = 0$$

$$(1-\lambda) \left( 1 + (1-\lambda)(3-\lambda) \right) = 0$$

$$(1-\lambda) \left( 1 + 3 - 4\lambda + \lambda^2 \right) = 0$$

$$(1-\lambda) \left( 4 - 4\lambda + \lambda^2 \right) = 0$$

$$(1-\lambda)(2-\lambda)^2 = 0, \quad \Rightarrow \begin{array}{l} \lambda=1, \text{ multiplicity } 1 \\ \lambda=2, \text{ multiplicity } 2 \end{array}$$

→ We don't know, we need to check basis of null space for  $\lambda=2$

$$\underline{A} - 2\underline{I} = \begin{pmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ -1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} -1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{array}{l} 2 \text{ free variables} \\ \rightarrow \dim \text{ null} = 2 \\ \Rightarrow \underline{A} \text{ is diagonalizable.} \end{array}$$

eg.  $\underline{\underline{A}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ , diagonalizable?

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = \lambda^2 + 1 = 0, \lambda = \pm i$$

- this is fine.

- diagonalizable!

$$\underline{\underline{D}} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \underline{\underline{A}} - i \underline{\underline{I}} = \begin{pmatrix} -i & 1 \\ -1 & i \end{pmatrix} \sim \begin{pmatrix} -i & 1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{aligned} -ix_1 + x_2 &= 0 & x_1 &= ix_2 \\ x_2 &= 1 & x_2 &= 1 \end{aligned}$$

$\Rightarrow \begin{pmatrix} i \\ 1 \end{pmatrix}$  is an eigenvector of  $\underline{\underline{A}}$  w/ eigenvalue  $i$ .