

Last time

Determinants (of square matrices)

1) $\det(\underline{A}) = 0 \iff \underline{A}^{-1}$ exists,

2) $\det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = 1 \det \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} = -3$

4) Row operations

- adding row to another → no change
- scaling row by $c \rightarrow c \cdot \det$
- swap rows → $-\det$

This time

Eigenvalues / Eigenvectors

$$\underline{A} \text{ square, if } \underline{A} \underline{x} = \lambda \underline{x}$$

↑ eigenvalue
↑ eigenvector

→ \underline{A} applied to \underline{x} does not change its direction
just scales it

Perhaps worthily, this is probably the most important thing from this course

- EVERYTHING (pretty much) is eigenvalues

$$\text{eg. } \underline{\underline{A}} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \underline{\underline{v}}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \underline{\underline{v}}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\underline{\underline{A}} \underline{\underline{v}}_1 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \text{ so } \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ is eigenvector of eigenvalue 3}$$

$$\underline{\underline{A}} \underline{\underline{v}}_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \text{ so } \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ is eigenvector of eigenvalue -1.}$$

$$\text{eg. } \underline{\underline{A}}^{90} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \underline{\underline{A}} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \text{ then}$$

$$\begin{aligned} \underline{\underline{A}}^{90} \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \underline{\underline{A}}^{89} \cdot 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \underline{\underline{A}}^{88} \cdot 3^2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= 3^{90} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3^{90} \\ 3^{90} \end{pmatrix} \end{aligned}$$

How to find them? Determinants!

Eigenvalue: $\underline{A}\underline{x} = \lambda\underline{x}$

$$\rightarrow \underline{A}\underline{x} - \lambda\underline{x} = \underline{0}, (\underline{A} - \lambda\underline{I})\underline{x} = \underline{0}$$

\rightarrow For a non-trivial \underline{x} to satisfy this equation

$\rightarrow \underline{A} - \lambda\underline{I}$ cannot have linearly independent columns

$$\rightarrow \det(\underline{A} - \lambda\underline{I}) = 0$$

this is called the characteristic polynomial

$p(\lambda)$ of \underline{A} .

- VERY IMPORTANT

- The roots (zeroes) of this polynomial are the eigenvalues of \underline{A}

eg. $\underline{\underline{A}} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \underline{\underline{A}} - \lambda \underline{\underline{I}} = \begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix}$

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = (1-\lambda)(1-\lambda) - 4 = 0$$

$$\Rightarrow 1 - 2\lambda + \lambda^2 - 4 = 0, \quad \lambda^2 - 2\lambda - 3 = 0$$

$$\Rightarrow (\lambda - 3)(\lambda + 1) = 0$$

$\Rightarrow \lambda = 3, \lambda = -1$ are the eigenvalues.

eg. $\underline{\underline{A}} = \begin{pmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{pmatrix}, \quad \underline{\underline{A}} - \lambda \underline{\underline{I}} = \begin{pmatrix} 1-\lambda & 4 & 5 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 3-\lambda \end{pmatrix}$

$$\begin{aligned} \det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) &= (1-\lambda) \det \begin{pmatrix} 2-\lambda & 6 \\ 0 & 3-\lambda \end{pmatrix} \\ &= (1-\lambda)(2-\lambda)(3-\lambda) = 0, \end{aligned}$$

$$\Rightarrow \lambda = 1, \lambda = 2, \lambda = 3.$$

How to find eigenvectors?

e.g. $\underline{A} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

$$\lambda = 3, \quad \underline{A - 3I} = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$$

$$\sim \begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix} \rightarrow \begin{array}{l} 2x_1 = 2x_2 \\ x_2 = x_2 \end{array}$$

$$\underline{Ax} = \lambda \underline{x} \quad (\text{eigenvalue eq.})$$

$$\rightarrow (\underline{A - \lambda I}) \underline{x} = \underline{0}$$

↑ eigenvector

→ eigenvectors are null space
of $\underline{A - \lambda I}$

→ basis null $\underline{A - 3I} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$, which is basis for eigenvectors
of $\lambda = 3$.

$$\lambda = -1, \quad \underline{A - (-1)I} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} \rightarrow \begin{array}{l} 2x_1 = -2x_2 \\ x_2 = x_2 \end{array}$$

→ basis null $\underline{A - 2I} = \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$, which is eigenvector basis
for $\lambda = -1$.

Eigenvalues can be complex:

e.g. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$,

$$\det(A - \lambda I) = \det \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 1 = 0$$

$$\lambda^2 = -1, \quad \lambda = i \\ \lambda = -i$$

→ This is valid! Actually extremely common to see.

Why?

1) Diagonalisation of matrices

2) Growth rates - stability of systems

3) Singular value decomposition

- data compression.

- I will show you an example.

- I will expand upon