

last time

$$T: V \rightarrow W$$

↑

$$\mathcal{B}_W = \{\underline{w}_1, \dots, \underline{w}_m\}$$

$$\mathcal{B}_V = \{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$$

Find matrix of  $T$  w.r.t. these bases  $\mathcal{B}_V$  and  $\mathcal{B}_W$ .

$$\Rightarrow \left( \begin{bmatrix} T(\underline{v}_1) \\ \vdots \\ T(\underline{v}_n) \end{bmatrix}_{\mathcal{B}_W} \quad \cdots \quad \begin{bmatrix} T(\underline{v}_n) \\ \vdots \\ T(\underline{v}_1) \end{bmatrix}_{\mathcal{B}_W} \right)$$

Example

$$T: \mathbb{P}^3 \longrightarrow \mathbb{P}^2$$

↑                      ↑

$$\mathcal{B} = \{1, x, x^2, x^3\} \quad C = \{1, x, x^2\}$$

$$T: f(x) \mapsto x \frac{d^2 f}{dx^2}$$

$$T(1) = x \cdot \frac{d^2}{dx^2}(1) = 0$$

$$T(x) = x \cdot \frac{d^2}{dx^2}(x) = 2x$$

$$T(x^2) = x \cdot \frac{d^2}{dx^2}(x^2) = 6x^2$$

$$[T(1)]_C = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$[T(x)]_C = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

⇒

$$[T(x^2)]_C = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$$

$$[T(x^3)]_C = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$$

$$\Rightarrow \text{matrix is } \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix}$$

- Is it onto? **No.** No pivot in each row

- Is it one-to-one? **No.** No pivot in each column.

Rank theorem: rank + nullity = # cols

$$= \# \text{ pivot} + \# \text{ free} = 2+2=4, \text{ still works as usual.}$$

Q) Find basis for nullspace, range( $T$ ).

A) Same as usual:

Column Space:  $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix}$  → Basis Range( $T$ ) =  $\{2x_1, 6x_2\}$

pivot columns

Nullspace:  $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix} \Rightarrow \begin{array}{l} x_1 = x_1 \\ x_2 = x_2 \\ 2x_3 = 0 \\ 6x_4 = 0 \end{array} \Rightarrow x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

→ Basis Null( $T$ ) =  $\{1, x\}$ .

# New topic: Determinants (Only defined for square matrices)

2x2:  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , determinant =  $ad - bc$

How to  
compute  
in general?

e.g.  $\begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 4 & 0 & -1 \end{pmatrix}$

- 1) Write sign matrix

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

2) Pick a row or column (any row/column) e.g. 1st row

$$\begin{pmatrix} 1+ & 2- & 1+ \\ -1 & 0 & 2 \\ 4 & 0 & -1 \end{pmatrix}$$

← assign signs

3) Expand upon row, using

$$1 \cdot \det \begin{pmatrix} 0 & 2 \\ 0 & -1 \end{pmatrix} - 2 \cdot \det \begin{pmatrix} -1 & 2 \\ 4 & -1 \end{pmatrix} + 1 \cdot \det \begin{pmatrix} -1 & 0 \\ 4 & 0 \end{pmatrix}$$

*cross out row and column to get submatrix*

$$\begin{pmatrix} 1+ & 2- & 1+ \\ -1 & 0 & 2 \\ 4 & 0 & -1 \end{pmatrix} \quad \begin{pmatrix} -1+ & 2- & 1+ \\ -1 & 0 & 2 \\ 4 & 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 1+ & 2- & 1+ \\ -1 & 0 & 2 \\ 4 & 0 & -1 \end{pmatrix}$$

$$= 1 \cdot 0 - 2 \cdot -7 + 1 \cdot 0$$

$$= 14$$

→ Does not matter which row/col you pick

→ Pick row/col of most zeroes

$$\begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 4 & 0 & -1 \end{pmatrix} \quad \underline{\text{Why?}}$$

$$= -2 \left| \begin{matrix} -1 & 2 \\ 4 & -1 \end{matrix} \right| + 0 \underbrace{\left| \begin{matrix} 1 & 1 \\ 4 & -1 \end{matrix} \right|}_{=0} - 0 \underbrace{\left| \begin{matrix} 1 & 1 \\ -1 & 2 \end{matrix} \right|}_{=0}$$

↑  
 best row  
 to pick

$$= 14, \quad \underline{\text{way easier.}}$$

For even larger matrices, same deal:

e.g.

$$\begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -2 & 1 & 0 \end{pmatrix} \rightarrow -1 \cdot \det \begin{pmatrix} 0 & 2 & 3 \\ 1 & 4 & 0 \\ -2 & 1 & 0 \end{pmatrix} - 0 \cdot \det(\text{something}) = 0$$

$$+ 0 \cdot \det(\text{something}) - 1 \cdot \det \underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & -2 & 1 \end{pmatrix}}_{=0} = 0$$

$$= -1 \cdot 3 \cdot \det \begin{pmatrix} 1 & 4 \\ -2 & 1 \end{pmatrix} - 1 \cdot 1 \cdot \det \begin{pmatrix} 1 & 4 \\ -2 & 1 \end{pmatrix}$$

$$= -1 \cdot 3 \cdot 9 - 1 \cdot 1 \cdot 9 = -36.$$

This is quite tedious! Some things to make it easier.

Theorem 1) If one row of  $A$  is added to another row to create  $B$ ,  
 $\rightarrow \det(A) = \det(B)$

e.g.  $\begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix} \xrightarrow{R_2 - 2R_1} \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix}$

$\det = 0$   $\det = 0$

2) If swap rows,  $\det$  swaps signs

e.g.  $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \xrightarrow{\text{swap rows}} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$\det = -3$   $\det = 3$

3) If multiply a row by  $k$ ,  $\det$  multiplies by  $k$

e.g.  $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \xrightarrow{R_1 \cdot 2} \begin{pmatrix} 2 & 4 \\ 2 & 1 \end{pmatrix}$

$\det = -3$   $\det = -6$

$\rightarrow$  Can now reduce to make this easier

FACT:

Diagonal matrix :  $\begin{pmatrix} a_1 & & & 0 \\ & a_2 & & \\ 0 & & \ddots & \\ & & & a_n \end{pmatrix}$ ,  $\det = a_1 a_2 \dots a_n$

You should prove yourself as an exercise.

Upper triangular :  $\begin{pmatrix} a_{11} & a_{12} & \cdots & \\ a_{22} & a_{23} & \cdots & \\ 0 & a_{33} & \cdots & a_{nn} \end{pmatrix}$ ,  $\det = a_{11} a_{22} \dots a_{nn}$

Lower triangular :  $\begin{pmatrix} a_{11} & & & 0 \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ & & & a_{nn} \end{pmatrix}$ ,  $\det = a_{11} a_{22} \dots a_{nn}$

IN GENERAL, Unless there three types, the determinant is NOT the product of the diagonal.

IDEA: Row reduce to triangular & calculate determinant

$$\text{eg: } \underline{\underline{A}} = \begin{pmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{pmatrix}$$

$$\det \underline{\underline{A}} = \left| \begin{array}{cccc} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{array} \right| = 2 \left| \begin{array}{cccc} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{array} \right|$$

$$= 2 \left| \begin{array}{cccc} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{array} \right| = 2 \left| \begin{array}{cccc} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{array} \right|$$

$$= 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & 2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 2 \cdot (1 \cdot 3 \cdot -6 \cdot 1) = -36.$$

Why do we care?

1)  $\det(\underline{\underline{A}}) \neq 0 \iff \underline{\underline{A}}$  is invertible

2) Eigenvalues (this week material)

- Possibly most important topic because

Properties

1)  $\det(\underline{\underline{A}}) = \det(\underline{\underline{A}}^\top)$

2) If  $\underline{\underline{A}}, \underline{\underline{B}}$  square,  $\det(\underline{\underline{AB}}) = \det(\underline{\underline{A}})\det(\underline{\underline{B}})$

One application of determinants:

## Cramer's Rule

replace  $i$ th column  
of  $\underline{A}$  with  $\underline{b}$

For  $\underline{A}\underline{x} = \underline{b}$ ,  $\underline{A}$  invertible. Then  $x_i = \frac{\det(\underline{A}_i(\underline{b}))}{\det(\underline{A})}$

e.g.

$$\begin{aligned} 3x_1 - 2x_2 &= 6 \\ -5x_1 + 4x_2 &= 8 \end{aligned}$$

$$\Rightarrow \underline{A} = \begin{pmatrix} 3 & -2 \\ -5 & 4 \end{pmatrix}, \det = 2$$

$\rightarrow$  invertible

$$\underline{A}_1(\underline{b}) = \begin{pmatrix} 6 & -2 \\ 8 & 4 \end{pmatrix}$$

$$\underline{A}_2(\underline{b}) = \begin{pmatrix} 3 & 6 \\ -5 & 8 \end{pmatrix}$$

$$\det = 24 + 16 = 40$$

$$\det = 24 + 30 = 54$$

$$\Rightarrow x_1 = \frac{40}{2} = 20$$

$$x_2 = \frac{54}{2} = 27$$

$$\Rightarrow \underline{x} = \begin{pmatrix} 20 \\ 27 \end{pmatrix}$$

→ How often do this used? Not really ever.