

Last Time:  $B = \{\underline{b}_1, \underline{b}_2, \dots, \underline{b}_n\}$  basis for  $V_1, \underline{v} \in V_1$

$$\underline{v} = c_1 \underline{b}_1 + c_2 \underline{b}_2 + \dots + c_n \underline{b}_n$$

$$\Leftrightarrow [\underline{v}]_B = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

$$\Leftrightarrow \underline{v} = \underbrace{\left( \underline{b}_1, \underline{b}_2, \dots, \underline{b}_n \right)}_{P_B} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \quad [\underline{v}]_B$$

If given 2 bases  $C, B$  of  $V_1$

$$\begin{array}{ccc} [\underline{x}]_C & \xrightarrow{P_C} & \underline{x} & \xrightarrow{P_B^{-1}} & [\underline{x}]_B \\ \xleftarrow{P_C^{-1}} & & & \xleftarrow{P_B} & \end{array}$$

$$\Rightarrow [\underline{x}]_C = \underset{C \Leftarrow B}{P} [\underline{x}]_B = P_C^{-1} P_B [\underline{x}]_B$$

$$[\underline{x}]_B = \underset{B \Leftarrow C}{P} [\underline{x}]_C = P_B^{-1} P_C [\underline{x}]_C$$

$$\underset{C \Leftarrow B}{P} = \underset{B \Leftarrow C}{P}^{-1}$$

Another way of seeing this:

$$\begin{aligned}\underline{\underline{P}}_{C \leftarrow B} &= \begin{pmatrix} P_C^{-1} P_B(\underline{e}_1) & \dots & P_C^{-1} P_B(\underline{e}_n) \end{pmatrix} \\ &= \begin{pmatrix} P_C^{-1}(\underline{b}_1) & \dots & P_C^{-1}(\underline{b}_n) \end{pmatrix} \\ &= \left( [\underline{b}_1]_C \quad \dots \quad [\underline{b}_n]_C \right)\end{aligned}$$

$= \underline{\underline{P}}_C^{-1} \underline{\underline{P}}_B$ , thus it the standard matrix  
for change of basis  $B \rightarrow C$ .

e.g.  $V = \mathbb{R}^2$ ,  $B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ ,  $C = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ .

$$[\underline{b}_1]_C = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ as } \underline{b}_1 = 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$[\underline{b}_2]_C = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \text{ as } \underline{b}_2 = -1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

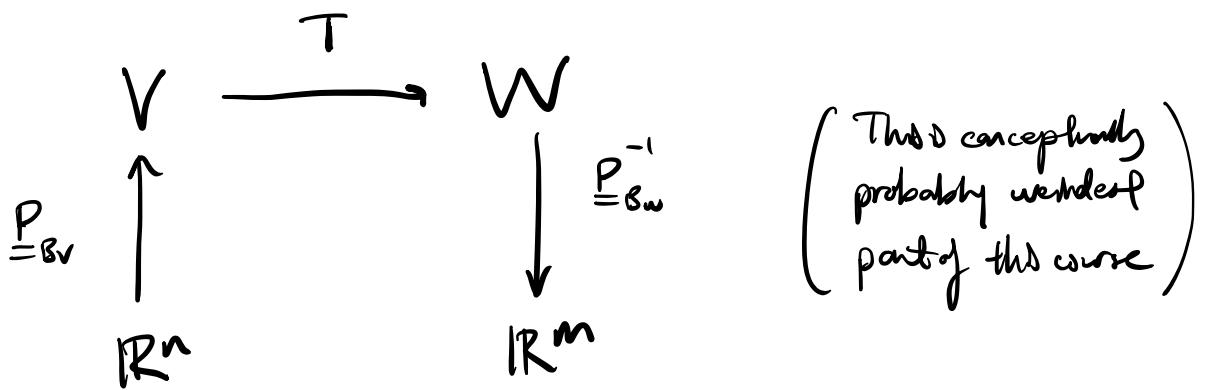
$$\Rightarrow \underline{\underline{P}}_{C \leftarrow B} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

- Another way of finding  
this.

## This time: Linear Transformations + Basis Sets

i.e.  $T: V \rightarrow W$ ,  $B_V = \{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$   
 linear  $B_W = \{\underline{w}_1, \underline{w}_2, \dots, \underline{w}_m\}$   
 bases for  $V$  and  $W$

Find matrix representation of  $T$  under these 2 bases:



$$\Rightarrow \text{matrix } \mathbf{P}_{B_W}^{-1} T \mathbf{P}_{B_V}$$

Standard Matrix:

$$\left( \mathbf{P}_{B_W}^{-1} T B_{B_V}(\underline{e}_1) \quad \dots \quad \mathbf{P}_{B_W}^{-1} T B_{B_V}(\underline{e}_n) \right)$$

$$= \left( \mathbf{P}_{B_W}^{-1} T(\underline{v}_1) \quad \dots \quad \mathbf{P}_{B_W}^{-1} T(\underline{v}_n) \right)$$

$$= \left( \begin{bmatrix} T(v_1) \\ T(v_n) \end{bmatrix}_{B_w} \dots \begin{bmatrix} T(v_n) \end{bmatrix}_{B_w} \right)$$

Example

$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$	$ $	$B = \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$
$T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 + 3x_2 \\ x_1 \end{pmatrix}$	$ $	$C = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \end{pmatrix} \right\}$

Find matrix of  $T$  relative to bases  $B$  and  $C$ .

i.e.  $TC[v]_B = [w]_C$

$\uparrow$  written  
in  $B$  basis       $\uparrow$  written in  
                         $C$  basis

Step 1: Send  $[v]_B \rightarrow v$ , standard basis.

We know  $P_B v_B = v$ ,  $P_B = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}$

Step 2: Transform in standard basis:

$$\underline{A} = (T(e_1) \ T(e_2)) = \begin{pmatrix} -1 & 3 \\ 1 & 0 \end{pmatrix}, \quad \underline{A} P_B v_B = \underline{A} v$$

Step 3: Send transformed result to C basis:

$$\text{We know } \underline{P}_c^{-1} \underline{x} = [\underline{x}]_c , \quad \underline{P}_c = \begin{pmatrix} 0 & 4 \\ 1 & -1 \end{pmatrix}$$

$$\underline{P}_c^{-1} = \frac{1}{4} \begin{pmatrix} 1 & 4 \\ 1 & 0 \end{pmatrix}$$

$$\text{so } \underbrace{\underline{P}_c^{-1} \underline{A} \underline{P}_B \underline{v}_B}_{\text{this is matrix of } T \text{ w.r.t. bases } B \text{ and } C.} = \underline{P}_c^{-1} \underline{A} \underline{v} = [\underline{w}]_c$$

↑

this is matrix of  $T$  w.r.t. bases  $B$  and  $C$ .

$$= \frac{1}{4} \begin{pmatrix} 1 & 4 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} 3 & 3 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 0 & 9 \\ 4 & 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 0 & \frac{9}{4} \\ 1 & \frac{1}{4} \end{pmatrix}}}$$

OR Use formula  $\left[ \left[ T(\underline{b}_1) \right]_c \left[ T(\underline{b}_2) \right]_c \right]$

$$T(\underline{b}_1) = \begin{pmatrix} 4 \\ -1 \end{pmatrix} , \quad T(\underline{b}_2) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\left[ T(\underline{b}_1) \right]_c = \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \quad \begin{pmatrix} 4 \\ -1 \end{pmatrix} = 0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$[T(b_2)]_c = \begin{pmatrix} \frac{9}{4} \\ \frac{1}{4} \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{9}{4} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

so we have  $\begin{pmatrix} 0 & \frac{9}{4} \\ 1 & \frac{1}{4} \end{pmatrix} = A_{B,C}$

Example  $T: \mathbb{P}_2 \rightarrow \mathbb{P}_1$ ,  $T(p) = p'$ , the derivative.

$$\begin{array}{ccc} \uparrow & & \uparrow \\ B = \{1, x, x^2\}, & & C = \{1, x\}, \text{ standard bases} \end{array}$$

$$= \left( [T(b_1)]_c \quad [T(b_2)]_c \quad [T(b_3)]_c \right)$$

$$= \left( [T(1)]_c \quad [T(x)]_c \quad [T(x^2)]_c \right)$$

$$= \left( [0]_c \quad [1]_c \quad [2x]_c \right)$$

$$= \left( \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \quad \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} \quad \begin{matrix} 0 \\ 0 \\ 2 \end{matrix} \right)$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Example  $T: P_2 \rightarrow P_1$ ,  $T$  is differentiation  
 $\uparrow \quad \uparrow$

$B = \{1, 2x, 4x^2 - 2\} = \{1, 2x, 4x^2 - 2\}$ , Hermite polynomials  
 - non-standard basis

Step 1: Send to standard basis

$$\underline{\underline{P}}_B [P]_B = P, \quad \underline{\underline{P}}_B = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Step 2: Transform in standard basis:  $\{1, x, x^2\}$

$$\underline{\underline{A}} = \begin{pmatrix} T(1) & T(x) & T(x^2) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \text{ from above}$$

Step 3: Transform back to  $B$

$$\underline{\underline{P}}_B^{-1} = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$$

$$\begin{aligned} \underline{\underline{P}}_B^{-1} \underline{\underline{A}} \underline{\underline{P}}_B &= \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

$$\text{OR} \quad \left( \begin{bmatrix} T(b_1) \end{bmatrix}_B, \begin{bmatrix} T(b_2) \end{bmatrix}_B, \begin{bmatrix} T(b_3) \end{bmatrix}_B \right)$$

$$= \left( \begin{bmatrix} 0 \end{bmatrix}_B, \begin{bmatrix} 2 \end{bmatrix}_B, \begin{bmatrix} 8x \end{bmatrix}_B \right)$$

$$= \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} = \underline{\underline{A}}_{BB}$$