Recall

Basis of V → Linearly independent
→ Spans V

What is V in general? Vector Space!

V non-empty set of addition and scalar multiply defined such that

1) $\underline{u}+\underline{v} \in V$

2) $\underline{u}+\underline{v} = \underline{v}+\underline{u}$

3) $(\underline{u}+\underline{v})+\underline{w} = \underline{u}+(\underline{v}+\underline{w})$

4) $\underline{0} \in V$

5) For $\underline{v} \in V$, $-\underline{v} \in V$ s.t. $\underline{v}-\underline{v}=\underline{0}$

6) $c\underline{u} \in V$

7) $c(\underline{u}+\underline{v}) = c\underline{u}+c\underline{v}$

8) $(c+d)\underline{v} = c\underline{v}+d\underline{v}$

9) $c(d\underline{u}) = (cd)\underline{u}$

10) $1\underline{u} = \underline{u}$

Subspace Conditions

eg. $\mathbb{R}^n$, which we have been using this whole time

eg. Polynomials of degree at most n, eg. $n=2$, $\mathbb{P}_2$

$a_0 + a_1 x + a_2 x^2$, $a_0, a_1, a_2 \in \mathbb{R}$,

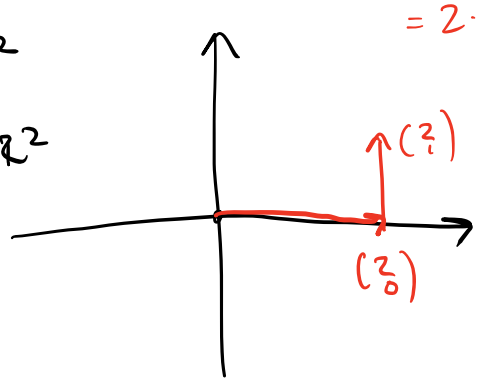
can check conditions are satisfied

Def Coordinate Systems

eg. $\mathbb{R}^2$

$\underline{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \in \mathbb{R}^2$

$= 2 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

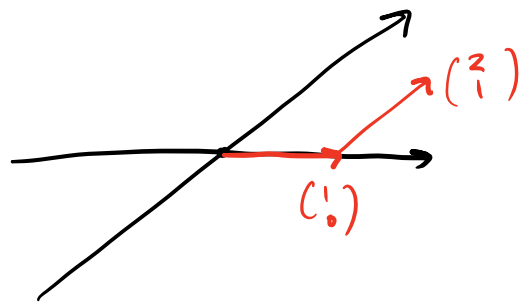


Standard Basis

$\mathcal{E} = \{ \underline{e}_1, \underline{e}_2 \}$

$[\underline{v}]_{\mathcal{E}} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$= 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

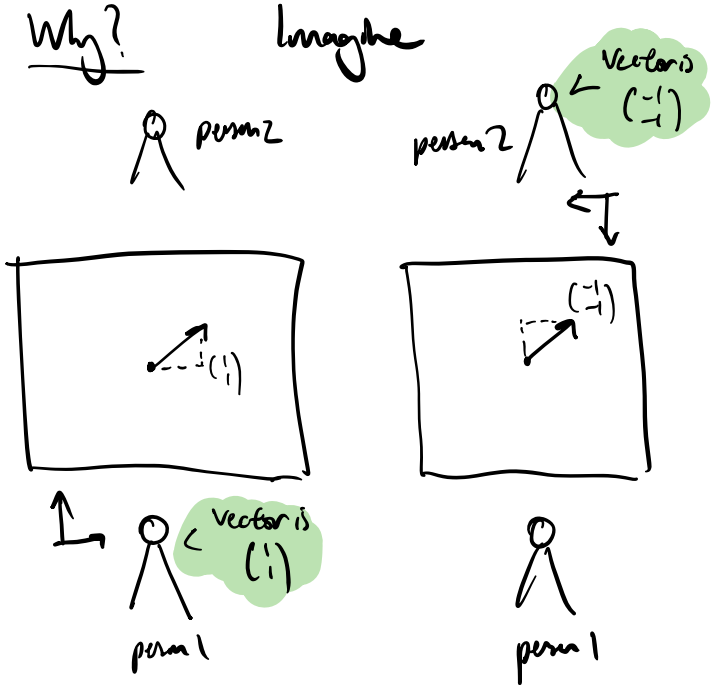


New Basis

$\mathcal{B} = \left\{ \underline{e}_1, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$

$[\underline{v}]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Why? Imagine



- Who is right?
  - Both! Different perspectives
- there 2 will describe the vector using different bases b/c they have different perspectives

How to find new coordinates in general?

- Row reduce!

eg. $V = \mathbb{P}_2$, $B_1 = \{1, x, x^2\}$, $B_2 = \{x, 1+x, x^2-1\}$

$x = 1 + 2x + x^2 \in V$,

$$[x]_{B_1} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \text{For } [x]_{B_2},$$

$$\alpha_1 x + \alpha_2 (1+x) + \alpha_3 (x^2 - 1) = 1 + 2x + x^2$$

$$\begin{aligned} 1: &\quad \alpha_2 - \alpha_3 = 1 \\ x: &\quad \alpha_1 + \alpha_2 = 2 \\ x^2: &\quad \alpha_3 = 1 \end{aligned} \iff \begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\longrightarrow \left(\begin{array}{ccc|c} 0 & 1 & -1 & 1 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right) \Rightarrow \begin{aligned} \alpha_1 &= 0 \\ \alpha_2 &= 2 \\ \alpha_3 &= 1 \end{aligned}, \quad [x]_{B_2} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.$$

<u>Theorem</u> There is a linear transformation from $\underline{x} \to [\underline{x}]_B$ given some basis B of V where $\underline{x} \in V$.

$\longrightarrow$ This transformation is an isomorphism.

<u>Proof</u> In book. Not that interesting.

<u>More interesting</u>: How do we write the matrix?

Let basis be $B = \{\underline{b}_1, \underline{b}_2, \ldots, \underline{b}_n\}$,

then $\underline{x} = c_1 \underline{b}_1 + \ldots + c_n \underline{b}_n$

$$= \underbrace{\begin{pmatrix} \underline{b}_1 & \cdots & \underline{b}_n \end{pmatrix}}_{\underline{P}_B} \underbrace{\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}}_{[\underline{x}]_B}$$

- Change of coordinate matrix from $B \to \mathcal{E}$

$$\begin{tikzcd} \underline{x} \arrow[r, bend left, "\underline{P}_B^{-1}"] & {[\underline{x}]_B} \arrow[l, bend left, "\underline{P}_B"] \end{tikzcd}$$

Change of Basis

- How to go from $[\underline{x}]_B \to [\underline{x}]_C$ in general?

IDEA:

$$\begin{tikzcd}
{[\underline{x}]_B} \arrow[r, bend left, "\underline{\underline{P}}_B"] & x \arrow[l, bend left, "\underline{\underline{P}}_B^{-1}"] \arrow[r, bend left, "\underline{\underline{P}}_C^{-1}"] & {[\underline{x}]_C} \arrow[l, bend left, "\underline{\underline{P}}_C"]
\end{tikzcd}$$

$$\underline{\underline{P}}_C [\underline{x}_C] = \underline{\underline{P}}_B [\underline{x}]_B$$

$$[\underline{x}_C] = \underbrace{\underline{\underline{P}}_C^{-1} \underline{\underline{P}}_B}_{\text{change of basis matrix } \underline{\underline{P}}_{C \leftarrow B}} [\underline{x}]_B$$

In general:

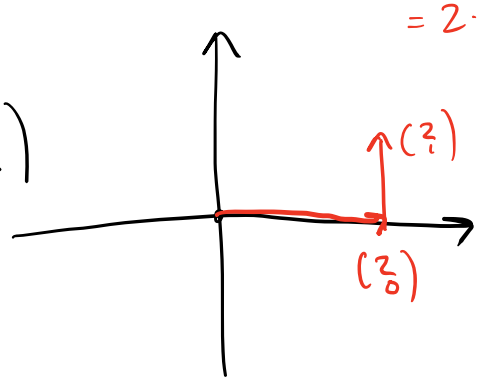
$$\underline{\underline{P}}_C^{-1} \underline{\underline{P}}_B = \underline{\underline{P}}_C^{-1} \left[\underline{b}_1 \; \cdots \; \underline{b}_n \right]$$

$$= \left[[\underline{b}_1]_C \; \cdots \; [\underline{b}_n]_C \right] = \underline{\underline{P}}_{C \leftarrow B}$$

eg.

$\underline{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$= 2 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



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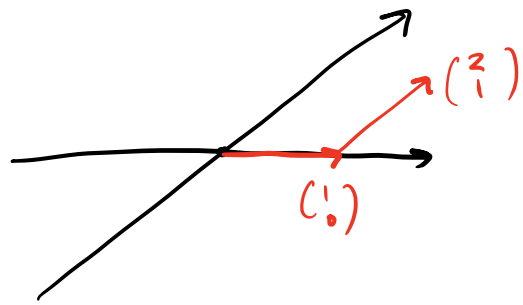


New Basis

$\mathcal{B} = \left\{ \underline{e}_1, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$

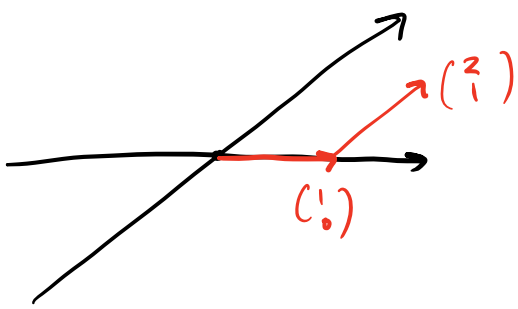
$[\underline{v}]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

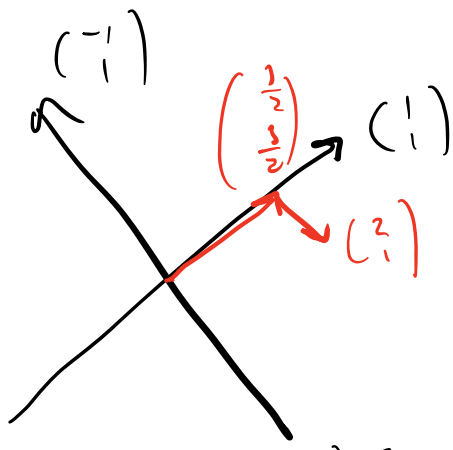
$\underline{\underline{P}}_{\mathcal{B}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, then $\underline{v} = \underline{\underline{P}}_{\mathcal{B}} [\underline{v}]_{\mathcal{B}}$

$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

eg.

$\underline{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$





$\mathcal{B}_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$

$[\underline{v}]_{\mathcal{B}_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\mathcal{B}_2 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$

$[\underline{v}]_{\mathcal{B}_2} = \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{pmatrix}$

$$\underline{\underline{P}}_{B} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \underline{\underline{P}}_{B_2} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\underset{B_2 \leftarrow B_1}{\underline{\underline{P}}} = \underline{\underline{P}}_{B_2}^{-1} \, \underline{\underline{P}}_{B_1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 1 \\ -\frac{1}{2} & 0 \end{pmatrix}$$

$$\underset{B_2 \leftarrow B_1}{\underline{\underline{P}}} [\underline{v}]_{B_1} = \begin{pmatrix} \frac{1}{2} & 1 \\ -\frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{pmatrix}$$

Calculate directly $\underset{B_2 \leftarrow B_1}{\underline{\underline{P}}}$!

$$\left(\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]_{B_2} \quad \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]_{B_2} \right) = \begin{pmatrix} \frac{1}{2} & 1 \\ -\frac{1}{2} & 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \end{array}\right) \qquad \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right)$$