

Last time

A  $n \times n$  has inverse  $\underline{A}^{-1}$

All  
equivalent  
for  $\underline{A} = n \times n$

- If has  $n$  linearly independent cols  $\rightarrow$  Equivalent for all  $\underline{A}$
- Has  $n$  pivot columns
- Has  $n$  pivot rows
- If columns  $\underline{A}$  span  $\mathbb{R}^n$   $\rightarrow$  Equivalent for all  $\underline{A}$
- Determinant is non-zero

This topic  
is not on  
midterm 1  
  
- You will need  
for midterm 2

### Determinants (How to calculate)

If  $\underline{A} = 2 \times 2$ ,  $\underline{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ,  $\det \underline{A} = |\underline{A}| = ad - bc$

For  $\underline{A} = 3 \times 3$ ,  $\underline{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

Steps: 1) Write alternating sign matrix

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

2) Pick a row or column, e.g. row 1

3) Expand along chosen row/column as follows

$$+a \underbrace{\begin{vmatrix} e & f \\ h & i \end{vmatrix}}_{\text{det of } 2 \times 2} - b \underbrace{\begin{vmatrix} d & f \\ g & i \end{vmatrix}}_{\text{det of } 2 \times 2} + c \underbrace{\begin{vmatrix} d & e \\ g & h \end{vmatrix}}_{\text{det of } 2 \times 2}$$

WHY?

$$\begin{pmatrix} a_+ & b_- & c_+ \\ d_- & e_+ & f_- \\ g_+ & h_- & i_+ \end{pmatrix} \rightarrow +a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix}$$

To expand across row:

$$\begin{pmatrix} a_+ & b_- & c_+ \\ d_- & e_+ & f_- \\ g_+ & h_- & i_+ \end{pmatrix} \rightarrow -b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix}$$

- Find sign of entry
- Cross out row + column
- Expand determinant of smaller system

$$\begin{pmatrix} a_+ & b_- & c_+ \\ d_- & e_+ & f_- \\ g_+ & h_- & i_+ \end{pmatrix} \rightarrow +c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$


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Add these up.

NOTE: It does not matter which row/column you expand on,

e.g.  $\underline{\underline{A}} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

If expand first row:

$$|\underline{\underline{A}}| = 1 \cdot \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + 3 \cdot \begin{vmatrix} 2 & 1 \\ 0 & 0 \end{vmatrix}$$

$$= 1 \cdot 1 - 2 \cdot 2 + 3 \cdot 0$$

$$= -3$$

If expand last row:

$$|\underline{\underline{A}}| = 0 \cdot \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} - 0 \cdot \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3$$

→ Expand row/column with the most zeros.

For bigger matrices, same thing:

e.g.  $\underline{A} = \begin{pmatrix} 1 & 4 & 2 & 0 \\ 2 & 0 & -1 & 0 \\ 3 & 1 & 2 & 4 \\ 1 & 0 & 1 & 0 \end{pmatrix}$

$\uparrow$   
expand this column

$\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$

$$= -4 \left| \begin{array}{ccc|c} 2 & -1 & 0 \\ 3 & 2 & 4 \\ 1 & 1 & 0 \end{array} \right| - 1 \left| \begin{array}{ccc|c} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 1 & 1 & 0 \end{array} \right|$$

$= -4 \left| \begin{array}{cc} 2 & -1 \\ 1 & 1 \end{array} \right| = 0, \text{ expand last column}$

$= 80$ . So the matrix  $\underline{A}$  is invertible.

- has lin. independent columns
- columns span  $\mathbb{R}^4$

Let's focus more on this  
for a second

Def

## Column Space of $\underline{\underline{A}}$

The span of the columns of  $\underline{\underline{A}}$  is known as the column space of  $\underline{\underline{A}}$ .

Def

## Null Space of $\underline{\underline{A}}$ (Also known as Kernel of $\underline{\underline{A}}$ )

The set of solutions to the homogeneous equation  $\underline{\underline{A}}\underline{x} = \underline{0}$  is the null space of  $\underline{\underline{A}}$ .

eg.  $\underline{\underline{A}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ , column space is  $\mathbb{R}^3$   
 - b/c column span  $\mathbb{R}^3$   
 null space  $\Rightarrow \{\underline{0}\}$

eg.  $\underline{\underline{A}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ , column space is  $\left\{ a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix}, a, b \in \mathbb{R} \right\}$   
 null space  $\Rightarrow \left\{ \begin{pmatrix} c \\ -c \end{pmatrix}, c \in \mathbb{R} \right\}$ .

## Def Subspace of $\mathbb{R}^n$

A subset  $H \subseteq \mathbb{R}^n$  is a subspace of  $\mathbb{R}^n$  if:

1)  $\underline{0} \in H$

2)  $\underline{u}, \underline{v} \in H \rightarrow \underline{u+v} \in H$

3)  $\underline{u} \in H \rightarrow c\underline{u} \in H$

e.g. Span of any set of vectors is a subspace of  $\mathbb{R}^n$

e.g.  $\text{Span} \left( \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \right)$

e.g.  $S = \left\{ \begin{pmatrix} a \\ 0 \end{pmatrix}, a > 0 \right\}$  is not a subspace

e.g.  $\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \notin S$

Prop: Column space and null space of  $\underline{A}^{(m \times n)}$  are subspaces of  $\mathbb{R}^m$ .

- Why? Because spans of vectors are .

Def. Basis set

Given a subspace  $S$  of  $\mathbb{R}^n$ , a set of linearly independent vectors

$V = \{v_1, v_2, \dots, v_n\}$  is a basis of  $S$  if  $\text{Span}(V)$  spans  $S$ .

Def. The number of vectors in a basis for  $S$ , is called its dimension.

\* The same number of vectors is always needed for a basis.

eg:  $S = \mathbb{R}^3$ .  $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$  is a basis of  $\mathbb{R}^3$ .  
dimension = 3

eg:  $S = \mathbb{R}^2$   $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ ,  $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ -3 \end{pmatrix} \right\}$   
one both bases for  $\mathbb{R}^2$ . dimension = 2

Def. The dimension of the column space of  $\underline{A}$  is called the rank of  $\underline{A}$ .

Def. The dimension of the null space of  $\underline{A}$  is called the nullity of  $\underline{A}$ .

### Theorem Rank - Nullity Theorem

If  $\underline{A}$  is  $m \times n$ , then  $\text{rank } \underline{A} + \text{nullity } \underline{A} = n$ .

# pivot columns      # free variables

e.g.  $\underline{A} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}$ ,  $\sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$

$\uparrow \uparrow \quad \uparrow$   
2 pivots      1 free

$$\Rightarrow \text{rank } \underline{A} = 2,$$

$$\text{basis column space } \underline{A} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$\text{nullity } \underline{A} = 2 \quad | \quad \text{basis null } \underline{A} = \left\{ \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \right\}$$