

Logistics

- Midterm next Friday
in class
- HW due 6pm

- Midterm pol
- Extra Credit next week

Some things from office hours / quiz:

$$1) \underline{A} \underline{x} = \begin{pmatrix} a_1 & a_2 & \dots & a_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = x_1 a_1 + x_2 a_2 + \dots + x_n a_n$$

- Linear combination of columns of \underline{A}

- If \underline{A} is $m \times n$, then columns of \underline{A} in \mathbb{R}^m .

2) $\mathbb{R}^2 \not\subseteq \mathbb{R}^3$. These are different spaces.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \quad \cancel{\rightarrow} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^3$$

You might think can just say

$$\rightarrow \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} \in \mathbb{R}^3 \quad X$$

BUT, why this as opposed to $\begin{pmatrix} 0 \\ x_1 \\ x_2 \end{pmatrix}$? Or $\begin{pmatrix} x_1 \\ 0 \\ x_2 \end{pmatrix}$?

- Thus, this not well defined. So \mathbb{R}^2 NOT subspace of \mathbb{R}^3 .

Last time

Matrix operations

1) Matrix addition

3) Matrix - matrix multiply

2) Scalar multiply

This time: Matrix Inverse

$$\underline{\underline{A}}^{-1} : \underline{\underline{A}}\underline{\underline{A}}^{-1} = \underline{\underline{A}}^{-1}\underline{\underline{A}} = \underline{\underline{I}}$$

Why? Say solving $\underline{\underline{A}}\underline{x} = \underline{b}$, $\underline{\underline{A}}^{-1}\underline{\underline{A}}\underline{x} = \underline{\underline{A}}^{-1}\underline{b}$

$\Rightarrow \underline{x} = \underline{\underline{A}}^{-1}\underline{b}$!!!, can solve equation
for any right hand side
without row reduction!

When does \underline{A}^{-1} exist?

Let's examine what it means, $\underline{x} = \underline{A}^{-1} \underline{b}$

→ Explicit formula, must be only one solution.

$$\underline{A} \quad \underline{x} = \underline{b}$$
$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ a_{31} & & & \\ \vdots & & & \\ a_{m1} & & & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

$m \times n$ $n \times 1$ $m \times 1$

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$\Rightarrow \begin{matrix} & & & & \\ & & & & \\ & & & & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m & & & & \end{matrix}$$

→ m equations, n unknowns.

- Matrix inverses can only exist when $m = n$.

~~Why?~~

$$\underline{\underline{A}}^{-1} \underline{\underline{A}} = \underline{\underline{A}} \underline{\underline{A}}^{-1} = \underline{\underline{I}}, \quad \underline{\underline{I}} \text{ is always square.}$$

$\begin{matrix} \uparrow & \uparrow \\ n \times m & m \times n \\ \curvearrowright & \\ n \times n & \end{matrix} \quad \begin{matrix} \uparrow & \uparrow \\ m \times n & n \times m \\ \curvearrowright & \\ n \times m & \end{matrix} \quad \rightarrow \quad \underline{\underline{A}} \text{ must be square.}$

 must be same.

\rightarrow If $\underline{\underline{A}}$ ^{$n \times n$} is square, and $\underline{\underline{A}}\underline{x} = \underline{b}$ has a unique solution, then $\underline{\underline{A}}^{-1}$ exists. When does this happen?

- When columns of $\underline{\underline{A}}$ are linearly independent!
- When columns of $\underline{\underline{A}}$ span \mathbb{R}^n !
 $\begin{matrix} \nearrow \text{equivalent b/c} \\ \underline{\underline{A}} \text{ is non} \end{matrix}$

Def

One-to-one transformation (Injective)

(Hopefully
reap)

$T: X \rightarrow Y$ transformation is one-to-one if for all $x_1 \neq x_2$

$$T(x_1) \neq T(x_2), \quad x_1, x_2 \in X.$$

Def

Onto transformation (Surjective)

$T: X \rightarrow Y$ transformation is onto if for all $y \in Y$, there is some $x \in X$ such that $T(x) =$

Def

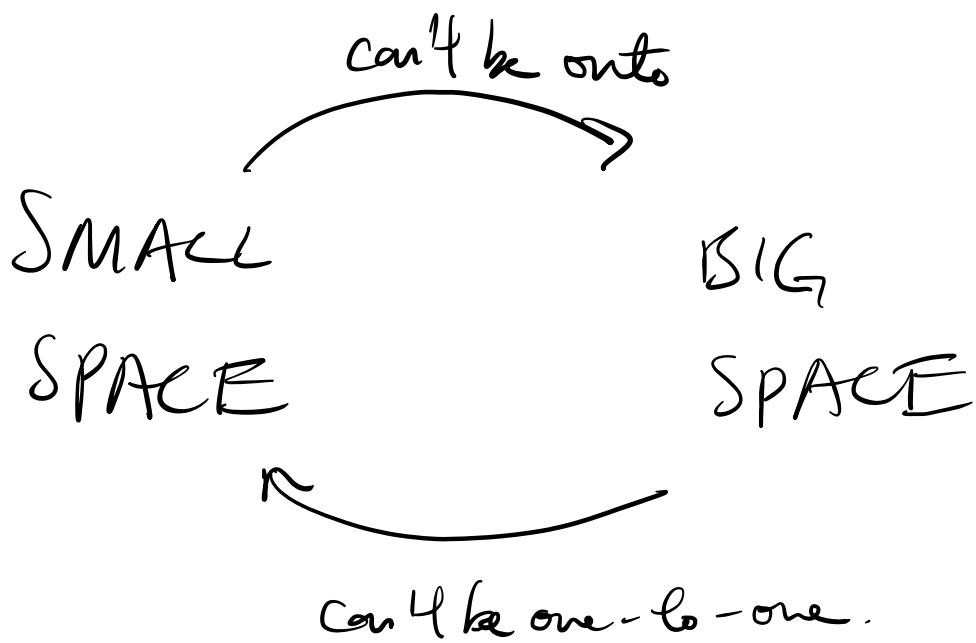
Bijective Transformation

$T: X \rightarrow Y$ transformation is a bijection if it is one-to-one and onto. (Only bijections have inverse)

e.g.

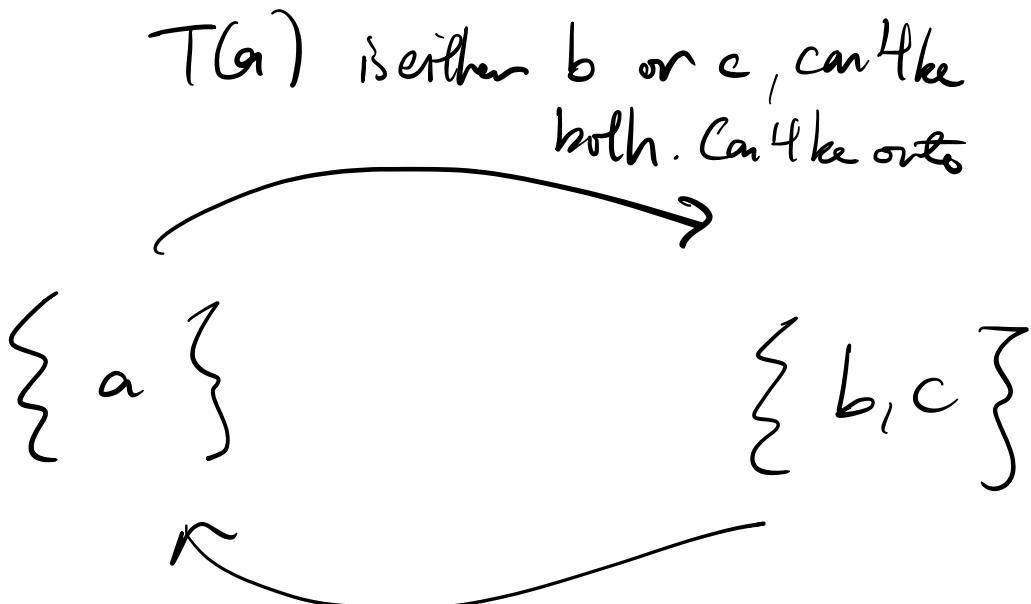
$T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$, $T\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$ is one-to-one and not onto.

eg. $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$, $T\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is onto but not one-to-one.



WHY?

eg.



$T(b) = T(c) = a$ must be true
(only one thing). Can't be one-to-one.

In the context of linear transformations: $T(\underline{x}) = \underline{\underline{A}} \underline{x}$

$T: \mathbb{R}^m \rightarrow \mathbb{R}^n$, so $\underline{\underline{A}}$ is $m \times n$

$$\begin{array}{ccc} \underline{\underline{A}} & \underline{x} & = \underline{b} \\ \uparrow & \uparrow & \uparrow \\ m \times n & \in \mathbb{R}^n, \text{ so } n \times 1 & \in \mathbb{R}^m, \text{ so } m \times 1 \end{array}$$

Case 1: $m > n$ e.g. $\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$, rows > cols
 - doesn't span \mathbb{R}^3
 → NOT onto!

Case 2: $n > m$ e.g. $\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$, cols > rows
 - can't have lin. independent columns!

Case 3: $n = m$, e.g. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, cols = rows
 (invertible)
 - can have lin. independent cols
 - can span \mathbb{R}^2

How to compute \underline{A}^{-1} ?

Easy case: 2×2

$$\underline{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \underline{A}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

why? $\frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$= \frac{1}{ad-bc} \begin{pmatrix} ad-bc & ab-bd \\ -ca+ac & -cb+ad \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$



this is called the determinant of \underline{A} . ($\det \underline{A}$)

Theorem

\underline{A} invertible if and only if $\det \underline{A} \neq 0$.

One way to find

Inverse \underline{A} in general:

$$\left(\begin{array}{c|c} A & I \\ \hline \equiv & \equiv \end{array} \right)$$

S

now reduce.

$$\left(\begin{array}{c|c} I & A^{-1} \\ \hline \equiv & \equiv \end{array} \right)$$

e.g.: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, A^{-1} = \frac{1}{4-6} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$

$$\rightarrow \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array} \right) \xrightarrow{R_2-3R_1} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array} \right)$$

$$\xrightarrow[R_2 \leftarrow R_2 - 2R_1]{R_1 \leftarrow R_1 - 2R_2} \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array} \right)$$

→ It works!

Are there other ways to find \underline{A}^{-1} ?

Yes. I will talk about more later.