Common Mistakes From HW

e.g. Determine if columns of $\underline{\underline{A}} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 2 & 0 & 4 \end{pmatrix}$ span $\mathbb{R}^3$ and if solution exists for every $\underline{y}$ to $\underline{\underline{A}}\underline{x} = \underline{y}$.

Mistake 1: $\underline{\underline{A}}$ is NOT an augmented matrix!!!

$\longrightarrow$ You can't talk about whether a system is consistent unless it is augmented!

i.e. $x_1 + x_2$, $2x_1 + 2x_2$, we can't say if it is consistent without knowing right hand side!

DON'T WRITE $\left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 2 & 0 & 4 \end{array}\right)$, the system isn't augmented!

INSTEAD, to see if it spans, you have to check if pivot in every row!

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 2 & 0 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow$$ No pivot in each row - doesn't span.

Mistake 2: Just because it doesn't span does not mean no solution for EVERY $\underline{y}$ to $\underline{\underline{A}}\underline{x} = \underline{y}$. It means no solution to SOME $\underline{y}$.

i.e. $\underline{\underline{A}}\underline{x} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 2 & 0 & 4 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$ has solution - consistent

$\underline{\underline{A}}\underline{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 2 & 0 & 4 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$ has no solution - inconsistent.

Basically $\underline{\underline{A}}\,\underline{x} = \underline{y}$ ONLY has solutions for $\underline{y} \in \text{Span}(\text{Columns of } \underline{\underline{A}})$.

This is because:

$$\begin{pmatrix} \underline{a}_1 & \underline{a}_2 & \cdots & \underline{a}_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} \longleftrightarrow \underbrace{x_1 \underline{a}_1 + x_2 \underline{a}_2 + \cdots + x_n \underline{a}_n}_{\text{linear combination of columns of } \underline{\underline{A}}} = \underline{y}$$

$= \underline{y}$

- This is why if that is each row for $\underline{\underline{A}} \in \mathbb{R}^{m \times n}$, then spans $\mathbb{R}^m$!!!

Recap

Last time, defined linear transformations $T(\underline{x})$, $\underline{x} \in \mathbb{R}^n$

and said $T(\underline{x}) = \underline{\underline{A}}\,\underline{x}$ for some $\underline{\underline{A}} = \begin{pmatrix} T(\underline{e}_1) & \cdots & T(\underline{e}_n) \end{pmatrix}$.

$\Rightarrow$ To understand linear transformation, can just understand matrices

This time, focus on what operations $\underline{\underline{A}}$ can do:

Vector-like operations:

1) Addition (must be same size)

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$$

2) Scalar multiplication

$$c\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} c & 2c \\ 3c & 4c \end{pmatrix}$$

Matrix – Matrix Multiplication

Recall: Matrix – Vector multiply

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \end{pmatrix}$$

Matrix – Matrix Multiply

– Just do the same, treat each column as vector

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} + 2a_{21} & a_{12} + 2a_{22} \\ 3a_{11} + 4a_{21} & 3a_{12} + 4a_{22} \end{pmatrix}$$

– each column of right hand side is just a matrix-vector multiply

RULE:

$$\underset{m \times p}{\overset{p}{\begin{pmatrix} \quad \end{pmatrix}}} \; \underset{p \times n}{\overset{n}{\begin{pmatrix} \quad \end{pmatrix}}} = \underset{m \times n}{\overset{n}{\begin{pmatrix} \quad \end{pmatrix}}}$$

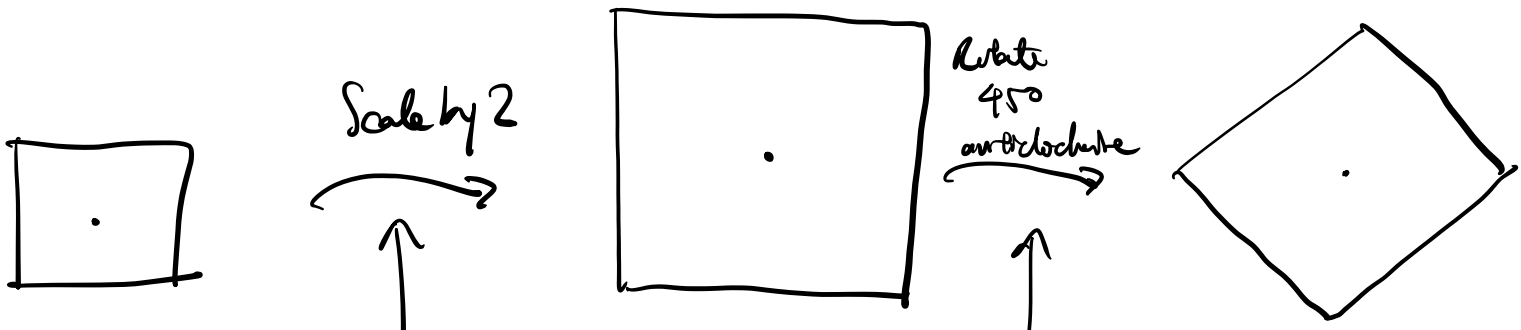
Can ONLY multiply if these are the same.

Why matrix-matrix multiply?

Remember that $\underline{\underline{A}}\,\underline{x}$ is a linear transformation on $\underline{x}$.

Then $\underline{\underline{B}}\,\underline{\underline{A}}\,\underline{x}$ is a transformation of $\underline{\underline{A}}\,\underline{x}$.

e.g. **Scale + Rotation**



$$\underline{\underline{A}} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad \underline{\underline{B}} = \begin{pmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{pmatrix}$$

- BUT, instead of calculating $\underline{\underline{A}}\,\underline{x}$, then $\underline{\underline{B}}(\underline{\underline{A}}\,\underline{x})$, can calculate $\underline{\underline{B}}\,\underline{\underline{A}}$, combining the transformations, then applying this compound transform to $\underline{x}$!

- WHY? In practice, want to apply compound transforms to more than one thing, this saves time.

Properties of mat-mat mul:

1) $(\underline{\underline{A}}\,\underline{\underline{B}})\underline{\underline{C}} = \underline{\underline{A}}(\underline{\underline{B}}\,\underline{\underline{C}})$, i.e. doesn't matter which order you compute in.

2) $\underline{\underline{A}}(\underline{\underline{B}} + \underline{\underline{C}}) = \underline{\underline{A}}\,\underline{\underline{B}} + \underline{\underline{A}}\,\underline{\underline{C}}$

3) $(\underline{\underline{A}} + \underline{\underline{B}})\underline{\underline{C}} = \underline{\underline{A}}\,\underline{\underline{C}} + \underline{\underline{B}}\,\underline{\underline{C}}$

4) $(c\underline{\underline{A}})\underline{\underline{B}} = c(\underline{\underline{A}}\,\underline{\underline{B}}) = \underline{\underline{A}}(c\underline{\underline{B}})$, c scalar

i.e. can move scalar around.

HOWEVER: IN GENERAL, $\underline{\underline{A}}\,\underline{\underline{B}} \neq \underline{\underline{B}}\,\underline{\underline{A}}$ (doesn't always commute)

eg. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 3 & 7 \end{pmatrix}$

NOT EQUAL

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 3 & 4 \end{pmatrix}$

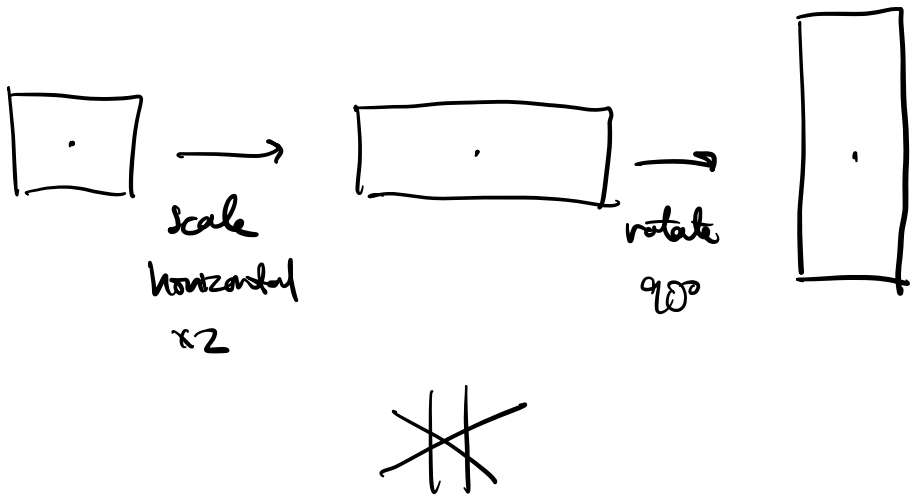
eg.

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 4 & 2 \end{pmatrix}$$

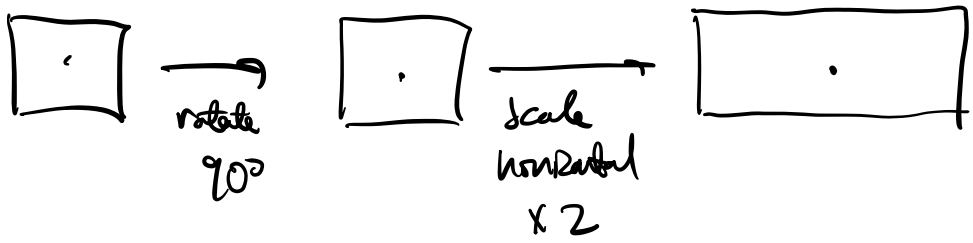
EQUAL

$$\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 4 & 2 \end{pmatrix}$$

In general, scale + rotate $\neq$ rotate + scale



$\neq$



— This is why matrix multiplication doesn't always commute.

Def. Identity matrix $\underline{\underline{I}}_n = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix}$, n 1's on diagonal, 0 everywhere else

e.g. $\underline{\underline{I}}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\underline{\underline{I}}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

– is the identity transformation (does nothing)

$\rightarrow$ $\underline{\underline{A}}\,\underline{\underline{I}}_n = \underline{\underline{A}}$, $\underline{\underline{I}}_n\,\underline{\underline{A}} = \underline{\underline{A}}$

e.g. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

Def. Matrix Transpose $\underline{\underline{A}}^T$.

Given $\underline{\underline{A}} = \begin{pmatrix} a_{11} & a_{12} & \\ a_{21} & & \ddots \end{pmatrix}$, $\underline{\underline{A}}^T = \begin{pmatrix} a_{11} & a_{21} & \cdots \\ a_{12} & & \ddots \\ \vdots & & \end{pmatrix}$ (swap columns and rows)

e.g. $\underline{\underline{A}} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, $\underline{\underline{A}}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

FACTS

$\rightarrow$ $(\underline{\underline{A}}^T)^T = \underline{\underline{A}}$ $\rightarrow$ $(c\underline{\underline{A}})^T = c\underline{\underline{A}}^T$

$\rightarrow$ $(\underline{\underline{A}} + \underline{\underline{B}})^T = \underline{\underline{A}}^T + \underline{\underline{B}}^T$ $\rightarrow$ $(\underline{\underline{A}}\,\underline{\underline{B}})^T = \underline{\underline{B}}^T\underline{\underline{A}}^T$

Why useful? I will tell you real time!