

Linear Transformations

$$\text{Recall: } \begin{aligned} x_1 + x_2 &= 1 \\ x_1 - x_2 &= 1 \end{aligned}$$

$$\rightarrow \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{\text{Matrix}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\text{Vector}} = \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\text{Vector}}$$

$$\text{Matrix equation: } \underline{A} \underline{x} = \underline{b}$$

→ Matrix-vector multiply, \underline{A} transforms \underline{x} into a vector \underline{b}

Notation: $\underline{x} \in \mathbb{R}^n$, $\underline{x} = \begin{pmatrix} : \\ : \\ : \end{pmatrix}^T$, n entries which are real numbers

$$\text{If } \underline{A} = \begin{pmatrix} & & \\ & & \\ \uparrow m & \uparrow & \downarrow n \\ & & \longleftarrow n \longrightarrow \end{pmatrix}, \quad \underline{x} = \begin{pmatrix} & \\ & \\ & \end{pmatrix}^T \begin{matrix} \uparrow \\ \downarrow \\ n \end{matrix}$$

$$\rightarrow \underline{A} \underline{x} = \begin{pmatrix} & \\ & \\ & \end{pmatrix}^T \begin{matrix} \uparrow \\ \downarrow \\ m \end{matrix}$$

In words, if $\underline{A} \in \mathbb{R}^{m \times n}$, $\underline{x} \in \mathbb{R}^n$, then $\underline{A}\underline{x} \in \mathbb{R}^m$.
 $m \times n$ real matrix

Equivalently, consider the:

vector transformation $T(\underline{x}) = \underline{A} \underline{x}$

Of course, not all transformations can be written as matrix-vector multiply.

e.g. $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, T(\underline{x}) = \begin{pmatrix} x_1^2 \\ x_2 \end{pmatrix}$ can't be written as matrix vector multiply

e.g. $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, T(\underline{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, can be,

$\underline{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \text{ as } \underline{A} \underline{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ in this case.

Q) So what types of transformations can be?

A) Linear Transformations!

Linear Transformations

Def. A transformation $T(\underline{x})$ is linear if it satisfies

$$1) \quad T(\underline{x} + \underline{y}) = T(\underline{x}) + T(\underline{y}) \quad , \quad \underline{x}, \underline{y} \text{ vectors}$$

$$2) \quad T(c\underline{x}) = cT(\underline{x}) \quad , \quad c \text{ scalar, } \underline{x} \text{ vector}$$

Example $T(\underline{x}) = 2\underline{x}$ is a linear transformation as

$$T(\underline{x} + \underline{y}) = 2(\underline{x} + \underline{y}) = 2\underline{x} + 2\underline{y} = T(\underline{x}) + T(\underline{y})$$

$$T(c\underline{x}) = 2(c\underline{x}) = c2\underline{x} = cT(\underline{x})$$

Example $\underline{x} \in \mathbb{R}^2, \quad T\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix}$

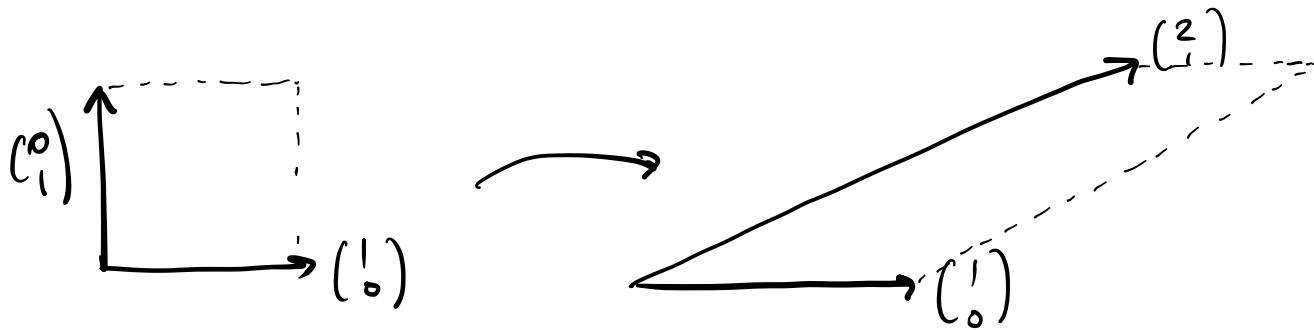
$$T(c\underline{x}) = T\left(\begin{pmatrix} cx_1 \\ cx_2 \end{pmatrix}\right) = \begin{pmatrix} c^2x_1^2 \\ c^2x_2^2 \end{pmatrix} = c^2 T(\underline{x}) \neq c T(\underline{x})$$

→ Transformation not linear

Physical Examples of Linear Transformations

1) Shear, let $T(x) = Ax$, $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

$$\text{Then } A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad A \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$



- People in engineering will see this A LOT

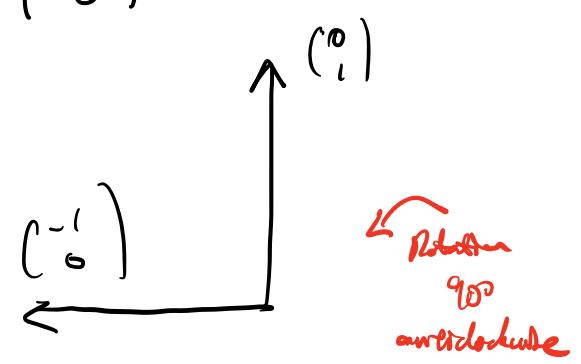
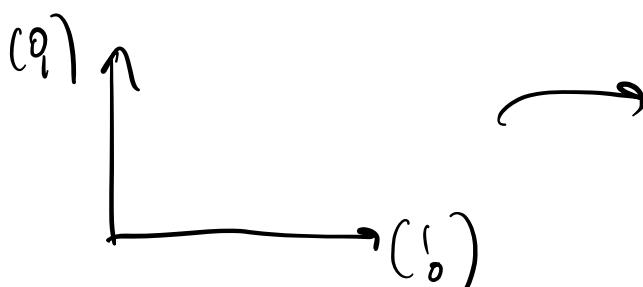
- Solid mechanics, elasticity

- Fluid mechanics, viscous fluids

2) Rotations $T(x) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} x$

Example $\theta = 90^\circ, \frac{\pi}{2}$ radians, $= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad A \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$



How to find matrix of a given linear transformation?

NOTATION $\underline{e}_i \in \mathbb{R}^n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ e.g. $\underline{e}_1 \in \mathbb{R}^2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \underline{e}_2 \in \mathbb{R}^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Any vector $\underline{x} \in \mathbb{R}^n \underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ can be written as the following linear combination

$$\underline{x} = x_1 \underline{e}_1 + x_2 \underline{e}_2 + \dots + x_n \underline{e}_n$$

e.g. $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Thus, $T(\underline{x}) = x_1 T(\underline{e}_1) + x_2 T(\underline{e}_2) + \dots + x_n T(\underline{e}_n)$

by linearity.

$$\Rightarrow \text{Matrix of } T = \begin{pmatrix} T(\underline{e}_1) & T(\underline{e}_2) & \dots & T(\underline{e}_n) \end{pmatrix}$$

e.g.: $T(x) = 2x$, $x \in \mathbb{R}^2$

$$T(e_1) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, T(e_2) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$\Rightarrow \underline{A} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \text{ check}$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} = 2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \quad \checkmark$$

e.g.: Rotation anticlockwise by θ

