Recap

Eg: $x_1 + x_2 = 1$
$x_1 - x_2 = -1$ ~ Matrix-Vector Form $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

- equivalent ways of writing

Definition Linear Independence

A set of vectors $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n\}$ is linearly independent if the only solution to

$$\alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 + \ldots + \alpha_n \underline{v}_n = 0$$

is the trivial solution $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$.

If the vectors are not linearly independent, they are linearly dependent.

eg: $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ is linearly independent.

$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\}$ is linearly dependent.

Theorem A set of vectors is linearly dependent if and only if one of the vectors in the set can be written as a linear combination of the others.

Proof Let $\underline{v}_k = \alpha_1 \underline{v}_1 + \dots + \alpha_{k-1} \underline{v}_{k-1} + \alpha_{k+1} \underline{v}_{k+1} + \dots + \alpha_n \underline{v}_n$

$$\Rightarrow 0 = \alpha_1 \underline{v}_1 + \dots + \alpha_{k-1} \underline{v}_{k-1} + 1 \cdot \underline{v}_k + \alpha_{k+1} \underline{v}_{k+1} + \dots + \alpha_n \underline{v}_n$$

non-zero coefficient

$\rightarrow$ Set is linearly dependent

$\rightarrow$ The converse is similar.

e.g. $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix} \right\}$, check if linearly dependent.

$$\begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \longleftrightarrow \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 1 & 1 & 5 \\ 0 & 1 & 2 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right)$$, system consistent, so

$\begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$ in span $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$. So they are linearly dependent.

Therefore: to check whether a set of vectors are linearly independent $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$, just need to check if

the matrix $\begin{pmatrix} \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_n \end{pmatrix}$ has pivot in every column.

e.g. $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ clearly pivot in each column

$\longrightarrow$ linearly independent.

e.g. $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, no pivot last column

$\longrightarrow$ linearly dependent.

e.g. $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 5 \\ 2 \end{pmatrix} \right\} \rightarrow \begin{pmatrix} 1 & 1 & 4 & 5 \\ 2 & 3 & 1 & 5 \\ 1 & 1 & 2 & 2 \end{pmatrix}$

clearly, there cannot be pivot in every column, as more columns than rows.

$\longrightarrow$ linearly dependent.

Why does this matter?

We know that linear independence means

$$\begin{pmatrix} \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_n \end{pmatrix}$$ has n pivot colums.

That means $$\begin{pmatrix} \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \underline{b},$$

always has unique solution for $\underline{\underline{\text{ANY}}}$ $\underline{b}$ if

$\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$ are linearly independent!