Last time

$$x + 2y = 2$$
$$x - y = -1$$
$$\longleftrightarrow \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 1 & -1 & -1 \end{array}\right)$$

We rewrote linear systems of equations as "augmented" matrix

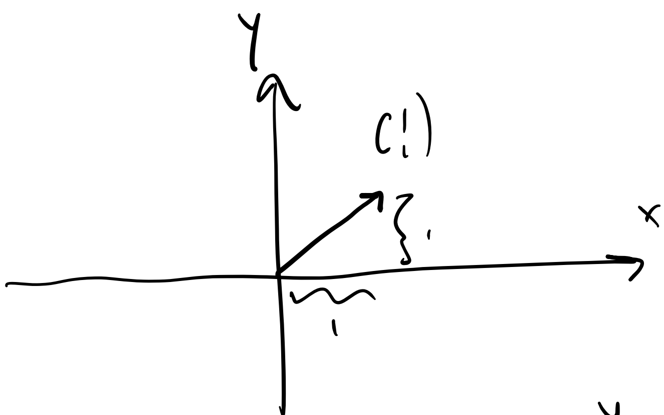
This time

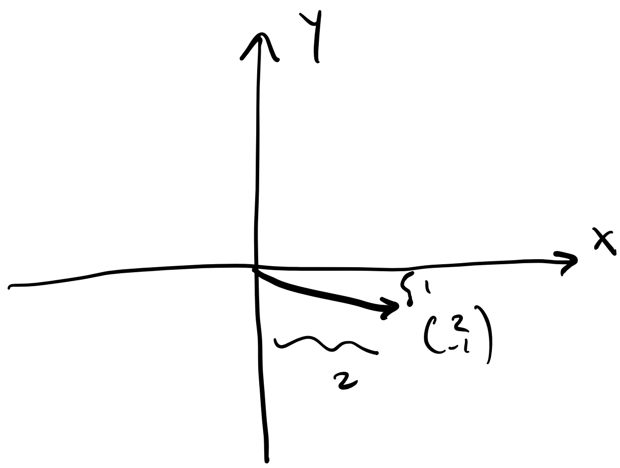
Write like this:

$$x\begin{pmatrix}1\\1\end{pmatrix} + y\begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}5\\-1\end{pmatrix}$$ "Vector Equation"

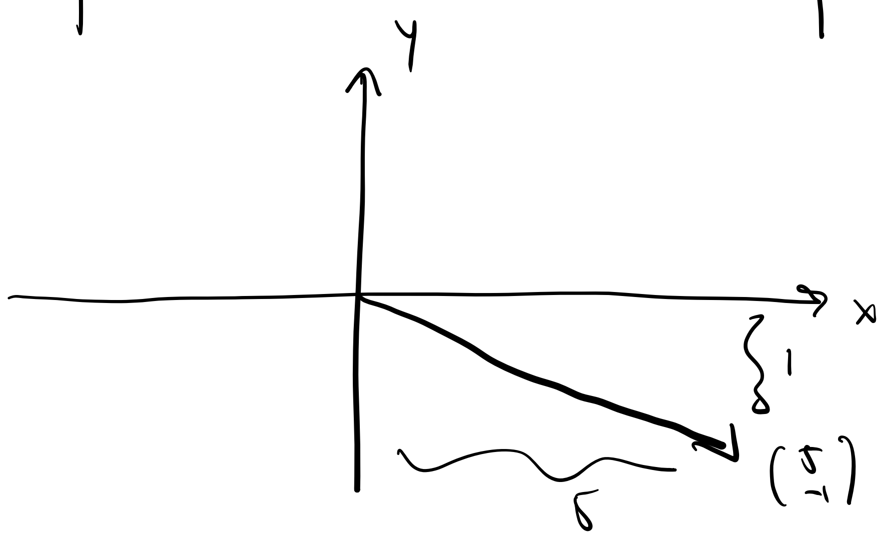
These are vectors

- How to understand the vectors? Geometrically is helpful:





PICTURES NOT TO SCALE

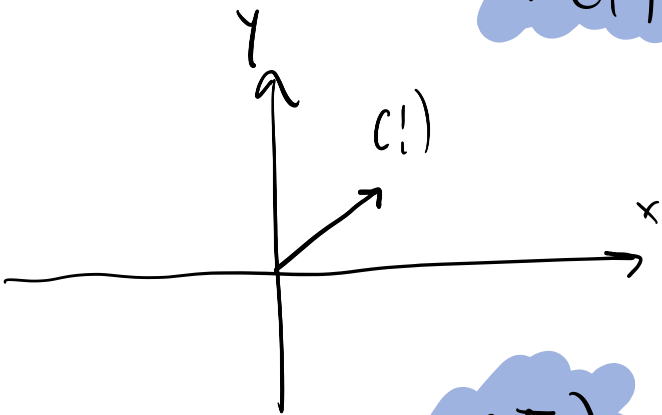


Row reduce to solve linear system:

$$\left(\begin{array}{cc|c} 1 & 2 & 5 \\ 1 & -1 & -1 \end{array}\right) \underset{R_2-R_1}{\sim} \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -3 & -6 \end{array}\right) \underset{\substack{R_2/-3 \\ \text{then } R_1-2R_2}}{\sim} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right)$$

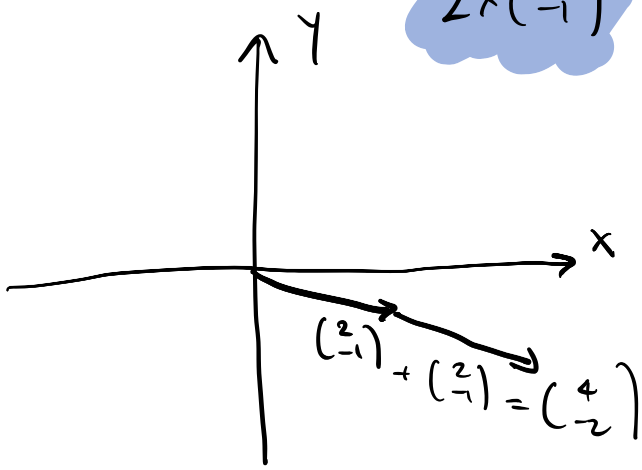
Geometrically

$1 \times \begin{pmatrix} 1 \\ 1 \end{pmatrix}$



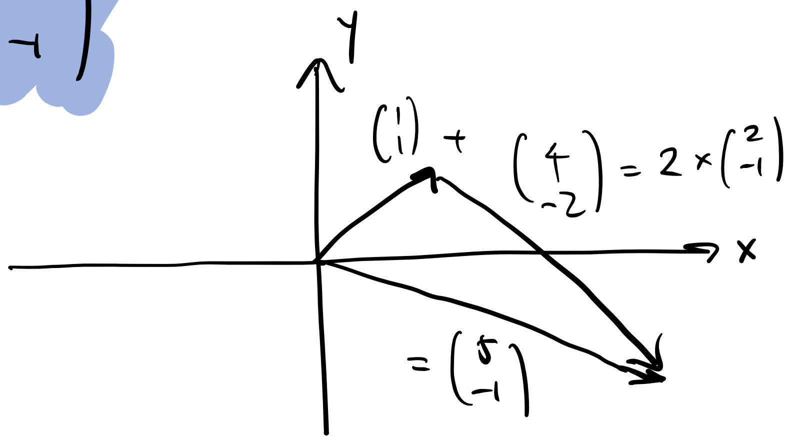
$+$

$2 \times \begin{pmatrix} 2 \\ -1 \end{pmatrix}$



$= \begin{pmatrix} 5 \\ -1 \end{pmatrix}$

ADD THEM UP $=$



$\Longrightarrow$ Can rephrase linear system

$$x + 2y = 5$$
$$x - y = -1$$

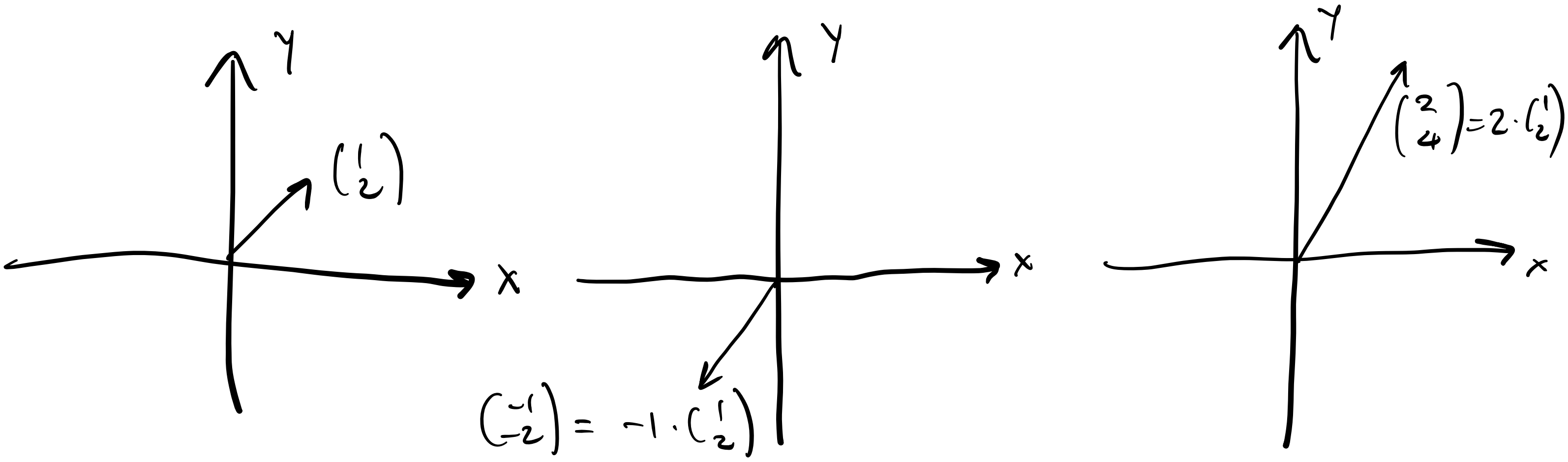
as how much of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ make $\begin{pmatrix} 5 \\ -1 \end{pmatrix}$

Vector operations (everything is componentwise)

Scalar Multiplication

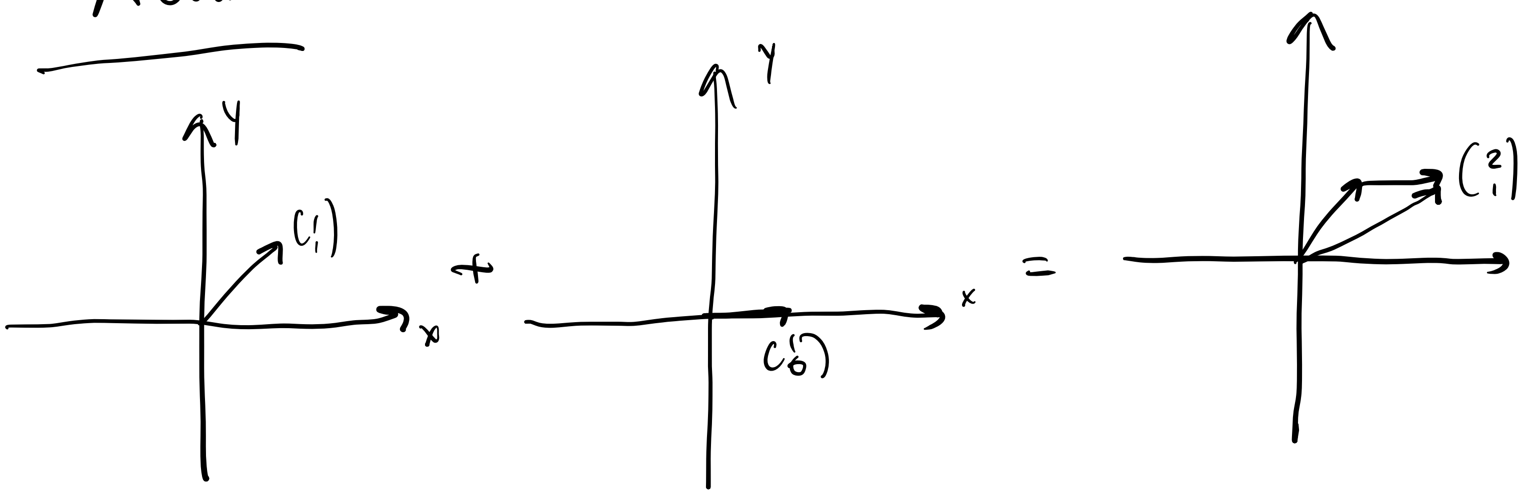
$$c\cdot\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}c\\2c\end{pmatrix}$$

c some number



Vector Addition

$$\begin{pmatrix}1\\1\end{pmatrix}+\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}$$



Definition A linear combination of vectors is the sum of scalar multiples of vectors

$$\underline{y} = \alpha_1 \underline{x}_1 + \alpha_2 \underline{x}_2 + \dots$$

where $\alpha_1, \alpha_2, \dots$ are numbers,
$\underline{x}_1, \underline{x}_2, \dots$ are vectors of the same size

e.g. $1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ is a linear combination of vectors

e.g. $2 \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 4 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 5 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is another valid lin. comb.

e.g. $\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ✗ NO.

– Only vectors of same size can be added

__Definition__ The span of a set of vectors $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$ is the set of vectors that can be written as a linear combination $\alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n$ for some scalars $\alpha_1, \alpha_2, \dots, \alpha_n$.

__eg:__ $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$ is in the span of $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\}$.

__eg:__ $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is in the span of $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

__eg:__ $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is __NOT__ in the span of $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

How to find if a vector is in the span of a set of vectors?

e.g. Is $\begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$ in span $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\}$?

→ This means is there α_1, α_2 such that

$$\alpha_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ -4 \end{pmatrix} ?$$

→ This is a linear system!

→ Row reduce!

$$\left(\begin{array}{cc|c} 1 & 0 & 5 \\ 2 & 2 & 2 \\ 0 & 1 & -4 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 2 & -8 \\ 0 & 1 & -4 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow \alpha_1 = 5, \ \alpha_2 = -4$. Check:

$$5\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - 4 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 10-8 \\ -4 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ -4 \end{pmatrix}, \text{ so yes!}$$

e.g.: Is $\begin{pmatrix}4\\0\\1\end{pmatrix}$ in span $\left\{\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\0\end{pmatrix}\right\}$?

$\Rightarrow$ Exist α_1, α_2 such that

$$\alpha_1\begin{pmatrix}0\\1\\1\end{pmatrix} + \alpha_2\begin{pmatrix}1\\2\\0\end{pmatrix} = \begin{pmatrix}4\\0\\1\end{pmatrix} ?$$

$$\rightarrow \left(\begin{array}{cc|c}0&1&4\\1&2&0\\1&0&1\end{array}\right) \sim \left(\begin{array}{cc|c}1&0&1\\1&2&0\\0&1&4\end{array}\right) \sim \left(\begin{array}{cc|c}1&0&1\\0&2&-1\\0&1&4\end{array}\right)$$

$$\sim \left(\begin{array}{cc|c}1&0&1\\0&1&-\frac{1}{2}\\0&1&4\end{array}\right) \sim \left(\begin{array}{cc|c}1&0&1\\0&1&-\frac{1}{2}\\0&0&\frac{9}{2}\end{array}\right) \sim \left(\begin{array}{cc|c}1&0&1\\0&1&-\frac{1}{2}\\0&0&1\end{array}\right)$$

pivot in solution column

$\rightarrow$ System inconsistent.

So it is not in the span.

ANOTHER WAY TO WRITE THE SYSTEM:

$$\begin{aligned} x + 2y &= 5 \\ 2x + y &= 3 \end{aligned} \longleftrightarrow \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & 1 & 3 \end{array}\right)$$

Equations

Augmented Matrix

$$\longleftrightarrow \quad x\begin{pmatrix} 1 \\ 2 \end{pmatrix} + y\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

Vector Equation

Matrix-Vector Multiplication

Matrix-Vector Form

Matrix-vector multiplication:

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

1st row × column = entry in 1st row

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

2nd row × column = entry in 2nd column

Matrix - Vector Multiplication

"A bit weird until you get used to it"

Matrix size

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

2 rows
3 columns $\Rightarrow$ 2x3 matrix

$$\begin{pmatrix} 1 & 0 \\ 4 & 0 \\ 3 & 1 \\ 5 & 0 \end{pmatrix}$$

4 rows
2 colms $\Rightarrow$ 4x2 matrix

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

3 rows
1 colm $\Rightarrow$ 3x1 matrix / 3 entry vector

(vectors are just 1 column matrices)

e.g.

$$\begin{pmatrix} 2 & 3 & 4 \\ 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

2x3 2x1

NOT DEFINED, these 2 MUST be the same

$$\begin{pmatrix} 2 & 2 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2\times1 + 2\times2 \\ 1\times1 + 4\times2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2\times1 + 2\times2 \\ 1\times1 + 4\times2 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ 9 \end{pmatrix}$$

e.g. $$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1\times2 + 0\times3 + 0\times4 \\ 0\times2 + 1\times3 + 0\times4 \\ 0\times2 + 0\times3 + 1\times4 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \text{ unchanged}$$

this is called the identity matrix as it does nothing to the vector