Practice Midterm 2 **Name:** Solutions
Date: 22/10/2021
Math 1A: Fall 2021

1. This practice is meant for 50 minutes - your exam will be 80 minutes.
2. As requested, this practice is designed to be pretty difficult.
3. I do not know what your actual exam looks like. The questions here are based on what I can gather from looking at previous midterms offered by Prof. Haiman + other professors.
4. The actual exam is closed book, no calculators. So for the best practice I would recommend doing that too for this.
5. You are allowed a single sided cheat sheet I believe for the exam - feel free to use it for this if you'd like.
6. Show your work. Don't just write down the answer. Answers with little justification will usually not get you many points.
7. If you want it graded, REMEMBER TO WRITE YOUR NAME on the top.

Score breakdown:

- 1: /10
- 2: /10
- 3: /10
- 4: /10
- 5: /10
- Total: /50

Problem 1 (10 Points)

Evaluate the limit

$$\lim_{x\to 0} \frac{\tan x}{3^x - 1}$$

$\tan(0) = 0$, $3^0 - 1 = 0$. So $\lim_{x\to 0} \frac{\tan x}{3^x - 1} \to \frac{0}{0}$

By l'hôpital's rule:

$$= \lim_{x\to 0} \frac{\sec^2(x)}{\ln(3)\cdot 3^x} = \frac{1}{\ln(3)}$$

Problem 2 (10 Points)

Use linear approximation or differentials to approximate $\ln(3)$. (Hint: Use $\ln(e)$)

$f(x) = \ln(x)$, $f(e) = \ln(e) = 1$, $f(3) = \ln(3)$

$f'(x) = \frac{1}{x}$, $f'(e) = \frac{1}{e}$

Linear Approx. Line: $y = \frac{1}{e}x + c$, $(x,y) = (e,1)$

$\Rightarrow 1 = \frac{e}{e} + c$, $c = 0$

So $\ln(3) \cong \frac{1}{e} \cdot 3 = \frac{3}{e}$

OR Using differentials: $dx = 3 - e$, $x = e$

$dy = \frac{1}{x} dx = \frac{(3-e)}{e}$

$\Rightarrow \ln(3) = \ln(e) + dy = 1 + \frac{(3-e)}{e}$

$= \frac{3}{e}$

Problem 3 (10 Points)

Differentiate the function $y = \sqrt{x}^{\sqrt{x}} e^{x^2}$.

Taking ln! $\ln y = \sqrt{x} \ln \sqrt{x} + x^2 \ln e$

$$\ln y = \frac{\sqrt{x}}{2} \ln x + x^2$$

Differentiate $\Rightarrow$ $\frac{y'}{y} = \frac{\sqrt{x}}{2} \cdot \frac{1}{x} + \frac{\ln x}{4\sqrt{x}} + 2x$

$$\frac{y'}{y} = \frac{1}{2\sqrt{x}} + \frac{\ln x}{4\sqrt{x}} + 2x$$

$$y' = \left(\frac{1}{2\sqrt{x}} + \frac{\ln x}{4\sqrt{x}} + 2x \right) y$$

$$= \left(\frac{1}{2\sqrt{x}} + \frac{\ln x}{4\sqrt{x}} + 2x \right) \sqrt{x}^{\sqrt{x}} e^{x^2}$$

Problem 4 (10 Points)

Find all local minima/maxima, and the global maximum and minimum of $f(x) = e^{|x|}\cos(x)$ on the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$. (Hint: Think symmetry.)

Symmetry: This function is even!

$$f(-x) = e^{|-x|}\cos(-x) = e^{|x|}\cos(x) = f(x).$$

So just need to check $x \in [0, \frac{\pi}{2}]$, where $f(x) = e^x \cos(x)$.

$$f'(x) = e^x\cos(x) - e^x\sin(x), \quad f'(x) = 0$$

$$\Rightarrow e^x\cos(x) = e^x\sin(x), \quad \cos(x) = \sin(x), \text{ so } x = \frac{\pi}{4}.$$

Also $f(x)$ is <u>not</u> differentiable at $x = 0$.

At $x=0$, e^x is increasing, as is $\cos(x)$. $\Rightarrow x = 0$, local min.

At $x = \frac{\pi}{4}$, $f''(x) = \cancel{e^x\cos(x)} - e^x\sin(x) - e^x\sin(x) \cancel{- e^x\cos(x)}$

$$f''(\tfrac{\pi}{4}) = -2e^{\frac{\pi}{4}}\sin(\tfrac{\pi}{4}) < 0.$$

$\rightarrow$ Local max.

Check endpoint for global max: $f(\frac{\pi}{2}) = 0$.

$\Rightarrow$ Global max. at $x = \frac{\pi}{4}, -\frac{\pi}{4}$.

Global min. at $x = -\frac{\pi}{2}, \frac{\pi}{2}$.

Problem 5 (10 Points)

The temperature of an experimental setup is given by the equation

$$T(t) = 100e^{-t}$$

Within the setup, the rate of some chemical reaction $k(T)$ is given by the equation

$$k(T) = T\sin\left(\frac{\pi T}{100}\right)$$

Find the change in rate of reaction in time when $t = 2$.

Want to find $\frac{dk}{dt}$ at $t=2$,

$$\frac{dk}{dt} = \frac{dk}{dT}\frac{dT}{dt}, \text{ by the Chain Rule}$$

$$T(2) = 100e^{-2}$$

$$\frac{dT}{dt} = -100e^{-t}, \quad t=2, \quad \frac{dT}{dt} = -100e^{-2}$$

$$\frac{dk}{dT} = \sin\left(\frac{\pi T}{100}\right) + T\cos\left(\frac{\pi T}{100}\right)\cdot\frac{\pi}{100}$$

$$\Rightarrow \frac{dk}{dt} = -100e^{-2}\left(\sin\left(\frac{\pi \cancel{100}e^{-2}}{\cancel{100}}\right) + 1\cancel{00}e^{-2}\cos\left(\frac{\pi\cancel{100}e^{-2}}{\cancel{100}}\right)\cdot\frac{\pi}{\cancel{100}}\right)$$

$$= \frac{-100}{e^2}\left(\sin\left(\frac{\pi}{e^2}\right) + \frac{\pi}{e^2}\cos\left(\frac{\pi}{e^2}\right)\right)$$