Practice Final **Name:** Solutions

Date: 8/12/2021

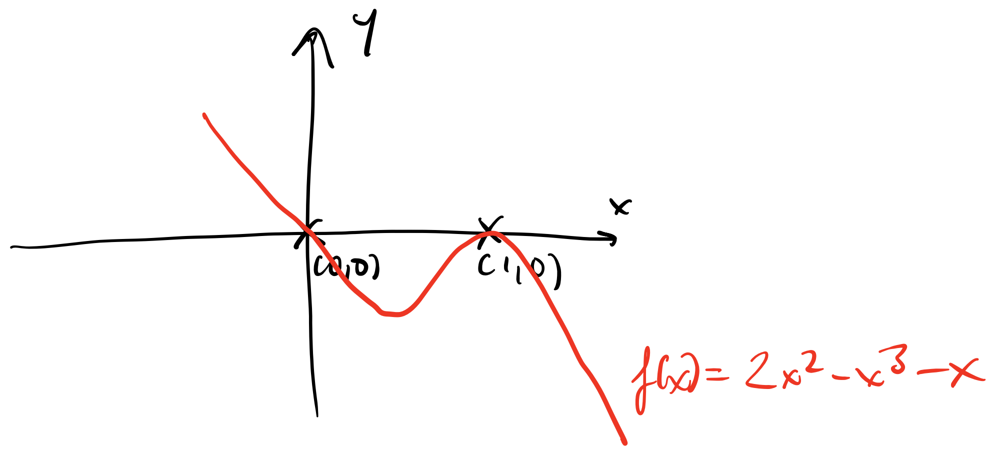
Math 1A: Fall 2021

1. Your real exam is 3 hours long. Sit this practice under the same conditions for the best practice.
2. I do not know what your actual exam looks like. The questions here are based on what I can gather from looking at previous midterms offered by Prof. Haiman + other professors.
3. The actual exam is closed book, no calculators. So for the best practice I would recommend doing that too for this.
4. You are allowed two single sided cheat sheets I believe for the exam - feel free to use it for this if you'd like.
5. Show your work. Don't just write down the answer. Answers with little justification will usually not get you many points.

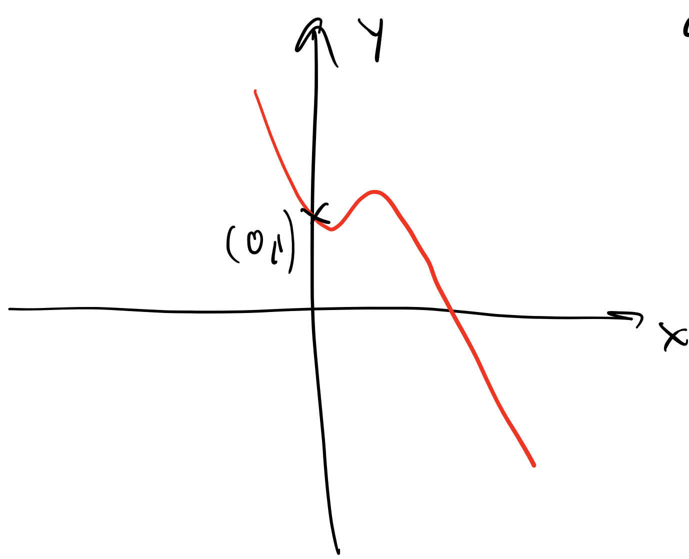
Problem 1 (10 Points)

1. Sketch the graph $f(x) = 2x^2 - x^3 - x$
2. Sketch the graph $f(2x)+1$. Describe the series of transformations from the first to the second.

1) $2x^2 - x^3 - x = -x(x-1)^2$



2)



a) Stretch horizontally factor $\frac{1}{2}$

b) Shift up 1

Problem 2 (10 Points)

Consider the function

$$f(x) = \begin{cases} \arctan(x) + \frac{\pi}{2},\ x < 0 \\ Ae^x + Bx + C,\ x \geq 0 \end{cases}$$

Find A, B, C such that the function is twice differentiable everywhere.

Continuity: $\lim_{x \to 0^-} \arctan(x) + \frac{\pi}{2} = \lim_{x \to 0^+} Ae^x + Bx + C$

$\implies \quad \frac{\pi}{2} = A + C$

1st Derivative Continuity: $\lim_{x \to 0^-} \frac{1}{1+x^2} = \lim_{x \to 0^+} Ae^x + B$

$\implies \quad 1 = A + B$

2nd Derivative Continuity: $\lim_{x \to 0^-} \frac{-2x}{(1+x^2)^2} = \lim_{x \to 0^+} Ae^x$

$\implies \quad 0 = A$

$\implies \quad B = 1$

$\implies \quad C = \frac{\pi}{2}$

Problem 3 (20 Points)

Find the following limits:

1. $\lim_{x\to 1} \frac{x^2+x+1}{x-1}$
2. $\lim_{x\to \pi/4} \frac{\cos^2(x)-\sin^2(x)}{e^x-1}$
3. $\lim_{x\to\infty} \sqrt{x^2+x}-x$
4. $\lim_{x\to-\infty} \sqrt{x^2+x}-x$

1) $\lim_{x\to 1^+} \frac{x^2+x+1}{x-1}$

Denominator is positive $\to 0$

Numerator is positive $\to 3$

$\Rightarrow \lim_{x\to 1^+} \frac{x^2+x+1}{x-1} = +\infty$

$\lim_{x\to 1^-} \frac{x^2+x+1}{x-1}$

Denominator is negative $\to 0$

Numerator $\to 3$

$\Rightarrow \lim_{x\to 1^-} \frac{x^2+x+1}{x-1} = -\infty$

$\Rightarrow \lim_{x\to 1} \frac{x^2+x+1}{x-1}$ Does not exist.

2) $\lim_{x\to \frac{\pi}{4}} \frac{\cos^2(x)-\sin^2(x)}{e^x-1}$

top $\to 0$

bot $\to e^{\frac{\pi}{4}}-1 \neq 0$

$\neq \infty$

$\Rightarrow \quad = 0.$

3) $\lim_{x\to\infty} \sqrt{x^2+x} - x \longrightarrow \infty - \infty$, indeterminate

$= \lim_{x\to\infty} \dfrac{x^2+x-x^2}{\sqrt{x^2-x}+x} = \lim_{x\to\infty} \dfrac{x}{\sqrt{x^2-x}+x}$

$= \lim_{x\to\infty} \dfrac{x}{\sqrt{x^2-x}+x} \dfrac{\cdot\frac{1}{x}}{\cdot\frac{1}{x}} = \lim_{x\to\infty} \dfrac{1}{\sqrt{1-\frac{1}{x}}+1}$

$= \dfrac{1}{2}$

4) $\lim_{x\to-\infty} \sqrt{x^2+x} - x \longrightarrow \underbrace{\sqrt{\infty - \infty}}_{\text{indeterminate}} + \infty$

$= \lim_{x\to-\infty} \dfrac{x}{\sqrt{x^2-x}+x}$, from above

$= \lim_{x\to-\infty} \dfrac{x}{\sqrt{x^2-x}+x} \dfrac{\cdot\frac{1}{x}}{\cdot\frac{1}{x}}$ $= -\sqrt{\frac{1}{x^2}}$ as $x\to-\infty$

$= \lim_{x\to-\infty} \dfrac{1}{-\sqrt{1-\frac{1}{x^2}}+1} = \infty.$

Problem 4 (10 Points)

1. Differentiate the function $f(x) = e^{e^x}$
2. Find the max/min of the function e^{-x^2} on the interval $[-1, 1]$.

1) $e^{e^x} = f_1(f_2(x))$

$f_1(x) = e^x \quad , \quad f_2(x) = e^x$

$\Rightarrow \frac{d}{dx} f_1(f_2(x)) = f_1'(f_2(x)) \cdot f_2'(x)$

$= e^{e^x} \cdot e^x$

2) $f(x) = e^{-x^2} \quad , \quad f'(x) = -2x\,e^{-x^2}$

$f(x)$ is everywhere differentiable.

Critical pts: $f'(x) = -2xe^{-x^2} = 0$

$\Rightarrow x = 0$.

For max/min, check critical pts + endpoints:

$x = -1 \quad , \quad f(x) = \frac{1}{e} \quad$, global min

$x = 0 \quad , \quad f(x) = 1 \quad$, global max

$x = 1 \quad , \quad f(x) = \frac{1}{e} \quad$, global min

Problem 5 (20 Points)

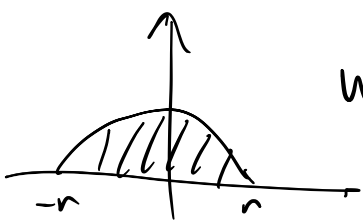
1. For some $r > 0$, find the definite integral $\int_{-r}^{r} \sqrt{r^2 - x^2} + x^3 dx$ in terms of r by interpreting it as an area.

2. Find the area bound by the curves $x = \frac{\pi}{4}, y = \tan x, y = 0$.

1) $\int_{-r}^{r} \sqrt{r^2-x^2} + x^3\, dx = \int_{-r}^{r} \sqrt{r^2-x^2}\, dx + \underbrace{\int_{-r}^{r} x^3 dx}_{=0,\ x^3 \text{ is odd}}$

$\int_{-r}^{r} \sqrt{r^2-x^2}\, dx$, $y = \sqrt{r^2-x^2}$, $y^2 = r^2 - x^2$

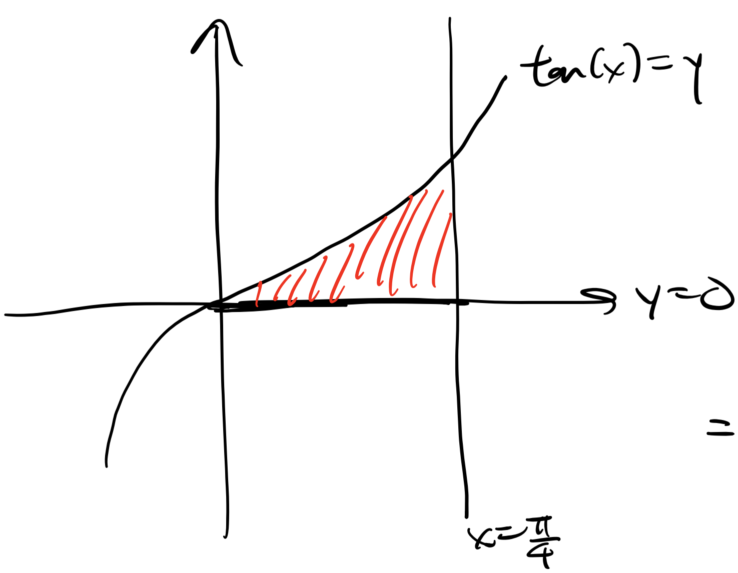
$\Rightarrow x^2 + y^2 = r^2$, this is a circle w/ radius r

$\Rightarrow$ 

Want this area $= \dfrac{\pi r^2}{2}$

$\Rightarrow$ Integral is $\dfrac{\pi r^2}{2}$

2) Sketch!



So area is $\int_0^{\frac{\pi}{4}} \tan(x) dx$

$= \int_0^{\frac{\pi}{4}} \frac{\sin(x)}{\cos(x)} dx$

$= -\ln|\cos x| \Big|_0^{\frac{\pi}{4}}$

$= -\ln|\cos\frac{\pi}{4}| + \ln|\cos 0|$

$= -\ln\left|\frac{1}{\sqrt{2}}\right| = \frac{\ln 2}{2}$.

Problem 6 (10 Points)

Using Newton's method approximate the value of $5^{\frac{1}{3}}$. Use a starting guess of $x_0 = 1$. Will it still work if $x_0 = 0$ instead?

Solve $f(x) = x^3 - 5$, as $f(5^{\frac{1}{3}}) = 0$.

Newton: $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$f'(x) = 3x^2$

$\Rightarrow x_1 = 1 - \dfrac{1-5}{3 \cdot 1} = 1 - \dfrac{-4}{3} = \dfrac{7}{3}$.

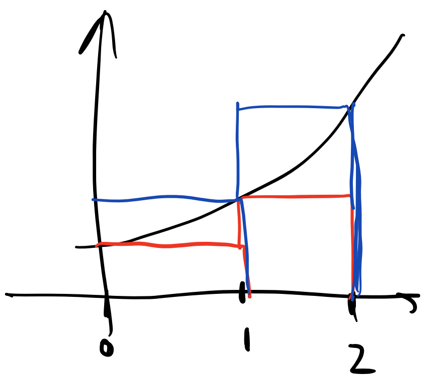
If $x_0 = 0$, it doesn't work as $f'(0) = 0$.

Problem 7 (20 Points)

1. Evaluate the integral $\int_0^2 e^x dx$.
2. Using Riemann sums with two blocks ($n = 2$), find two numbers B, C such that $B \leq \int_0^2 e^x dx \leq C$.
3. Find the volume of the solid formed by rotating the area bound by the curves $y = x^2, y = 2x$ about the line $x = 2$.

1) $\int_0^2 e^x dx = e^x \Big|_0^2 = e^2 - 1$

2)



left Riemann-Sum underestimates

right Riemann-Sum overestimates

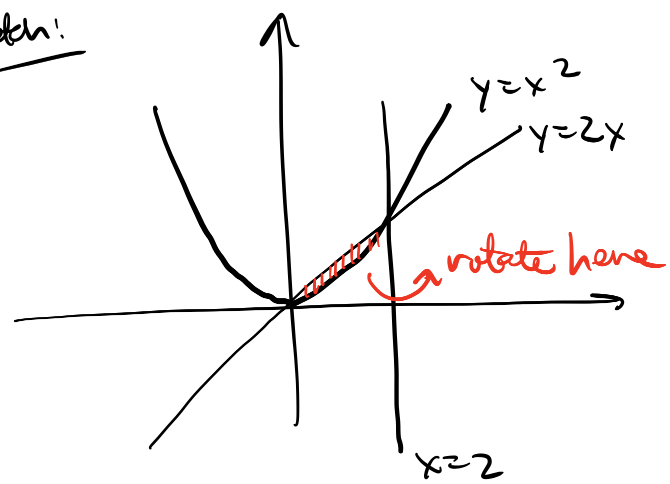
$\Rightarrow$ left: $e^0 + e^1$

right: $e^1 + e^2$

$\Rightarrow 1 + e \leq \int_0^2 e^x dx \leq e + e^2$

3) $y=x^2$, $y=2x$, rotate about $x=2$.

Sketch:



$y=x^2$, $y=2x$ intersection:

$x^2=2x$

$x^2-2x=0$, $x(x-2)=0$,

$x=0$, $x=2$.

$\Rightarrow$ Intersection points:

$(0,0)$, $(2,4)$

Using washer method:

$y=x^2 \Rightarrow x=\sqrt{y}$, $y=2x \rightarrow x=\frac{y}{2}$

$$\Rightarrow V=\int_0^4 \pi\left(2-\frac{y}{2}\right)^2-\pi\left(2-\sqrt{y}\right)^2 dy$$

Using cylindrical shells:

$$V=\int_0^2 2\pi(2-x)(2x-x^2)\, dx$$

Both give $V=\frac{16}{6}\pi$